Graduate Written Qualifying Exam

Day 2 – Modern Physics

Tuesday, August 23, 2011

*** Please read carefully before beginning ***

On day 2 of this exam you are asked to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

Many of the problems only need a few lines of calculation. If you find yourself in a lengthy calculation, stop and move on. If something appears unclear, don't hesitate to ask.

Good Luck!

Expressions, formulae, physical constants, integrals, etc
(which you may find useful although you may not need all of them)

Bose/Fermi distribution function: $\frac{1}{e^{(\epsilon-\mu)/k_B T}\mp 1}$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\oint_C f(z)\,dz = 2\pi i \sum \text{Res}\,[f] \qquad \ln(N!) \approx N \ln(N) - N, \quad \text{for } N \gg 1$$

$$\psi_{100}(\mathbf{r}) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \qquad \psi_{200}(\mathbf{r}) = \frac{1}{4\sqrt{2\pi a_0^3}}(2 - r/a_0)e^{-r/2a_0} \qquad \psi_{210}(\mathbf{r}) = \frac{1}{4\sqrt{\pi a_0^3}}(r/a_0)e^{-r/2a_0}\cos\theta$$

$$\int_0^\infty dx\, e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad \int_0^\infty dx\, xe^{-ax^2} = \frac{1}{2a} \qquad \int_0^\infty dx\, x^2 e^{-ax^2} = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$\int_0^\infty dx\, x\, e^{-ax} = \frac{1}{a^2} \qquad \int_0^\infty dx\, x^2\, e^{-ax} = \frac{2}{a^3} \qquad \int_0^\infty dx\, x^n\, e^{-ax} = \frac{n!}{a^{n+1}}$$

$$\epsilon_0 = 8.854 \times 10^{-12}\ \text{C}^2/\text{N m}^2 \qquad\qquad \mu_0 = 4\pi \times 10^{-7}\ \text{N s}^2/\text{C}^2$$

$$c = 1/\sqrt{\epsilon_0 \mu_0} = 3.0 \times 10^8\ \text{m/s} \qquad\qquad e = 1.602 \times 10^{-19}\ \text{C}$$

$$h = 6.626 \times 10^{-34}\ \text{J s} \qquad\qquad \hbar = h/2\pi = 6.582 \times 10^{-22}\ \text{MeV s}$$

$$\hbar c = 197\ \text{MeV fm} = 197\ \text{eV nm} \qquad\qquad (\hbar c)^2 = 0.389\ \text{GeV}^2\ \text{mbarn}$$

$$m_e = 0.511\ \text{MeV}/c^2 \qquad\qquad m_{proton} = 938\ \text{MeV}/c^2$$

$$1\ \text{eV} = 1.602 \times 10^{-19}\ \text{J} \qquad\qquad 1\ \text{eV}/c^2 = 1.783 \times 10^{-36}\ \text{kg}$$

$$k_B = 1.381 \times 10^{-23}\ \text{J/K} \qquad\qquad k_B = 8.617 \times 10^{-5}\ \text{eV/K}$$

Quantum Mechanics

In calculating electronic structure of ring-shaped molecules, one is led to consider simplified models. For example, take two identical fermions of mass m and spin 1/2, moving on a ring of radius a, with spins constrained to point *up*. The particles can interact through a potential of form

$$V(\phi_1, \phi_2) = A\cos(\phi_1 - \phi_2)$$

where ϕ_1 and ϕ_2 are angular positions of the two particles.

First assume A = 0

(a) Solve for the eigenfunctions and energies of this two particle system. [Hint: the only dynamics in this system are the constrained motion (kinetic energy) of a particle going in a circle. The Hamiltonian for each particle may therefore be written as a rigid rotator $\hat{H} = \hat{L}_z^2/2mr^2$]

(b) State the degeneracies of the ground and 1st excited states.

Now assume A is nonzero but weak, $A \ll \hbar^2/(2ma^2)$

(c) Find the energies and degeneracies of the ground state from (b), to first order in A.

Statistical Mechanics

Atoms on a solid surface can form a two dimensional ideal gas (as well as a wide variety of ordered phases). By controlling the temperature and vapor pressure above the surface, one can control the density of atoms on the surface (the two dimensional density is $n = N/A$, where N is the number of atoms on the surface and A is the area of the surface). The energy of an atom on the surface is

$$E = -\epsilon_0 + \sum_{i=1,2} \frac{p_i^2}{2m} \tag{1}$$

where ϵ_0 is the adsorption energy or binding energy holding the atom on the surface.

Reminders of quantities in three dimensions:

$$
\begin{aligned}
P &= \frac{N}{V} k_B T & (2) \\
\mu &= k_B T \ln[P/P_0(T)] & (3) \\
Z_1 &= V \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} = V \lambda^{-3}, & (4)
\end{aligned}
$$

where P is the vapor pressure, μ is the chemical potential, and Z_1 is the partition function for a single atom of mass, m. k_B is Boltzmann's constant and h is a constant with units of energy times time.

a) Using Eq. 1 for the energy, evaluate the partition function for one atom in a classical two dimensional ideal gas. Check your answer by comparing with Eq. 4.

b) What is the partition function for a classical two dimensional ideal gas of N identical particles?

c) From (b), evaluate the Helmholtz free energy; simplify your expression and show that it is extensive in the limit of large numbers of atoms.

d) What is the equation of state analogous to Eq. 2?

e) What is the chemical potential?

f) Finally, relate the three dimensional vapor pressure to the two dimensional density of atoms in equilibrium at temperature, T.

General Physics

Accretion of Planets

A number of recently discovered extrasolar planet candidates have surprisingly small orbits, which may indicate that considerable orbital migration takes place in protoplanetary systems. A natural consequence of orbital migration is for a series of planets to be accreted, destroyed, and then thoroughly mixed into the envelope of the central star. Stars may frequently swallow planets during the early phase of their evolution. The aim of this problem is to estimate the response of the star to such accretion.

The star has mass, M, and radius R (with the center of the star at $R = 0$) and constant density, such that $\rho = 3M/(4\pi R^3)$. Assume that a planet of mass $m << M$ and radius $r_p << R$ is accreted by the star from a circular orbit which just grazes the stellar surface.

(a) A planet of mass m in a circular orbit of radius R around a star of mass M has an orbital energy which is composed of both kinetic and potential energy. These two components obey a *Virial Equation*. State the virial equation for this circular orbit. What is the (total) orbital energy in terms of gravitational potential energy, Ω? Is the orbital energy of such a system positive or negative?

(b) As the planet is accreted by the star, the total energy (W) of the star increases by some amount ΔW. Let this amount be due to the orbital energy of the planet just as it is accreted. What is ΔW in terms of M, m, and R (and G the gravitational constant)?

(c) Derive the expression for gravitational potential energy of the star in terms of M and R. To do this you can write the gravitational potential energy for a spherical shell within the star. The shell is at radius r, has thickness dr, mass dM and encloses a mass $M(r)$. Write dm and $M(r)$ as a function of ρ and r and integrate over the star to determine to total Ω.

(d) Show that for a planet with properties similar to those of Jupiter (mass $10^{-3}M_\odot$, radius $0.1R_\odot$) and the star similar to the Sun, the dominant contribution to the change in the stellar energy comes from the planet's orbital energy rather than its gravitational binding energy (the negative of the gravitational potential energy).

(e) Once the star has accreted the planet, its mass has increased by m and its radius may have changed by some amount ΔR. By how much has its gravitational potential energy changed ($\Delta\Omega$) from what it was before the accretion? Derive an expression for $\Delta\Omega$ to first order in the small quantities m/M and $\Delta R/R$.

(f) A star in equilibrium also satisfies the Virial equation stated in (a). In the case of a star (which under most circumstances can be considered an ideal gas) the kinetic energy of a star is the same as its thermal energy. Using the thermodynamic relations between pressure and energy, i.e. γ-law relation, we can write the virial equation (which need not be derived) for our star such that

$$\Delta\Omega = \frac{3(\gamma - 1)}{3\gamma - 4}\Delta W ,\qquad\qquad (5)$$

where γ is the ratio of specific heats at fixed pressure and volume for the star.

(i) Using the virial equation above, what values of γ are allowed for a dynamically stable star (a star that neither explodes nor collapses)?

(ii) Use the virial equation above and your answer to (b), to solve for the fractional change in radius ($\Delta R/R$) in terms of m/M and γ.

(iii) Calculate $\Delta R/R$ that would be caused by the accretion of Jupiter (if $\gamma = 5/3$) onto the Sun. Does accretion cause the star to expand or contract? Why?