Graduate Written Qualifying Exam

Day 2 – Modern Physics

Feb. 2, 2010

***** Please read carefully before beginning *****

On day 1 of this exam you are asked to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

Many of the problems only need a few lines of calculation. If you find yourself in a lengthy calculation, stop and move on. If something appears unclear, don't hesitate to ask.

Good Luck!

Expressions, formula, physical constants, integrals, etc
(which you may find useful although you may not need all of them)

Bose/Fermi distribution function: $\frac{1}{e^{(\epsilon-\mu)/k_B T} \mp 1}$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\oint_C f(z)\,dz = 2\pi i \sum \text{Res}\,[f] \qquad \ln\big(N!\big) \approx N\,\ln(N) - N, \quad \text{for } N \gg 1$$

$$\psi_{100}(\mathbf{r}) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \qquad \psi_{200}(\mathbf{r}) = \frac{1}{4\sqrt{2\pi a_0^3}}(2 - r/a_0)e^{-r/2a_0} \qquad \psi_{210}(\mathbf{r}) = \frac{1}{4\sqrt{\pi a_0^3}}(r/a_0)e^{-r/2a_0}\cos\theta$$

$$\int_0^\infty dx\; e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad \int_0^\infty dx\; xe^{-ax^2} = \frac{1}{2a} \qquad \int_0^\infty dx\; x^2 e^{-ax^2} = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$\int_0^\infty dx\; x\,e^{-ax} = \frac{1}{a^2} \qquad \int_0^\infty dx\; x^2\,e^{-ax} = \frac{2}{a^3} \qquad \int_0^\infty dx\; x^n\,e^{-ax} = \frac{n!}{a^{n+1}}$$

$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N\,m^2}$ $\qquad$ $\mu_0 = 4\,\pi \times 10^{-7}\ \mathrm{N\,s^2/C^2}$

$c = 1/\sqrt{\epsilon_0 \mu_0} = 3.0 \times 10^8\ \mathrm{m/s}$ $\qquad$ $e = 1.602 \times 10^{-19}\ \mathrm{C}$

$h = 6.626 \times 10^{-34}\ \mathrm{J\,s}$ $\qquad$ $\hbar = h/2\pi = 6.582 \times 10^{-22}\ \mathrm{MeV\,s}$

$\hbar c = 197\ \mathrm{MeV\ fm} = 197\ \mathrm{eV\ nm}$ $\qquad$ $(\hbar c)^2 = 0.389\ \mathrm{GeV^2\ mbarn}$

$m_e = 0.511\ \mathrm{MeV}/c^2$ $\qquad$ $m_{proton} = 938\ \mathrm{MeV}/c^2$

$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$ $\qquad$ $1\ \mathrm{eV}/c^2 = 1.783 \times 10^{-36}\ \mathrm{kg}$

$k_B = 1.381 \times 10^{-23}\ \mathrm{J/K}$ $\qquad$ $k_B = 8.617 \times 10^{-5}\ \mathrm{eV/K}$

Quantum Mechanics

This problem concerns transitions from excited states of a hydrogen atom (2S and 2P) into the ground state (1S). Electron spin can be neglected for the purposes of this problem, and some orbital wave functions are given on the formula sheet. While answering the questions you may encounter various integrals. **Do not evaluate nontrivial integrals !** Instead, introduce clearly defined symbolic notations for their values.

(a) An atomic orbital couples to electromagnetic fields via

$$W = W_{DE} + W_{DM} + W_{QE} + \cdots$$

where

$$W_{DE} = -q\mathcal{E}Z, \quad W_{DM} = \frac{-q}{2m}\mathcal{B}L_x, \quad W_{QE} = \frac{-q}{2mc}\mathcal{E}(YP_z + ZP_y).$$

(i) For atomic transitions in general, which of the three terms is the most important, and why?

(ii) For each interaction, indicate if it allows or forbids a $2S \to 1S$ transition and explain why.

(iii) The interactions above omit the $q^2|\mathbf{A}|^2$ term from the quantum Hamiltonian $|\mathbf{p} - q\mathbf{A}|^2$. How does the $q^2|\mathbf{A}|^2$ term contribute to the $2S \to 1S$ transition?

(b) Explain why the 2P level is relatively short lived (its lifetime τ is about 10^{-10} seconds).

(c) The wavefunction

$$\psi(t) = b_\alpha(t)e^{-iE_\alpha t/\hbar}|\alpha\rangle + b_\beta(t)e^{-iE_\beta t/\hbar}|\beta\rangle$$

represents a superposition of states α=2S and β=2P. Their amplitudes vary as

$$i\hbar\dot{b}_\alpha = 0 \qquad i\hbar\dot{b}_\beta = -\frac{i\hbar}{\tau}b_\beta,$$

where we have used the long lifetime of 2S to approximate b_α as a constant, but we keep the short lifetime τ of 2P leading to decay of b_β. These equations represent the time-dependent Schrödinger equation in the subspace spanned by $\{\alpha, \beta\}$.

Now apply a constant electric field $\tilde{\mathbf{E}} = \mathcal{E}_0\hat{\mathbf{z}}$ to the atom and derive the new equations for $\dot{b}_\alpha$ and $\dot{b}_\beta$ in terms of matrix elements that you need not evaluate. Note that $\dot{b}_\alpha$ becomes nonzero and explain how this leads to a new mechanism for decay of the $2S$ state.

(d) Find the time constant for decay of 2S using your result of part (c). Since the energy difference $\Delta E_{\alpha\beta} = E_\alpha - E_\beta$ is rather small, you may simplify your calculations by setting $\Delta E_{\alpha\beta} = 0$. What is the 2S lifetime in the limit of strong applied electric field?

Statistical Mechanics

From classical electrodynamics we know – among many other things – the following facts about electromagnetic radiation:

- The radiation pressure p and the energy density $u = U/V$ of isotropic electromagnetic radiation are related by $p = \frac{1}{3}u$. (This is an "equation of state".)
- For *thermal radiation* this energy density $u = u(T)$ only depends on the temperature T.
- The number N of photons in thermal equilibrium cannot be fixed. Formally we can account for this fact by saying that their chemical potential is $\mu = 0$. (This, of course, goes beyond classical electrodynamics. Just bear with us.)

Based on these principles, you are now asked to derive an important result for thermal radiation.

1. Recall the identity $G = \mu N$ for the Gibbs free energy $G(T, p, N)$. Use it to show that the entropy of isotropic thermal radiation is given by $S = \frac{4}{3} V u(T)/T$.
2. Derive the Maxwell relation $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$.
3. Use the preceding two results to derive the well-known law $u = aT^4$ (with some undetermined integration constant a).

We will now look a bit into the quantum nature of the problem and derive the constant a from first principles.

4. The spectral (frequency-resolved) energy density $\tilde{u}(\omega)$ of thermal radiation is given by

$$\tilde{u}(\omega) = \frac{2}{V} \sum_{\vec{k}} \delta\left(\omega - \frac{\epsilon_{\vec{k}}}{\hbar}\right) \epsilon_{\vec{k}} \langle \hat{N}_{\vec{k}} \rangle \ . \tag{0}$$

   Here, $\epsilon_{\vec{k}}$ is the energy of a photon with wave vector $\vec{k}$, and $\langle \hat{N}_{\vec{k}} \rangle$ is the expected number of such photons at some prescribed temperature T; the overall prefactor 2 accounts for the two directions of polarization per photon.

   (a) Approximate the sum over $\vec{k}$ with a suitable integral, assuming a cubic box of length L.

   (b) Write down expressions (that you need not derive) for $\epsilon_{\vec{k}}$ and $\langle \hat{N}_{\vec{k}} \rangle$.
   Hint: recall one more fact about photons: they are bosons.

   (c) Derive an explicit formula for $\tilde{u}(\omega)$ in the thermodynamic limit $V \to \infty$.

5. By integrating $\tilde{u}(\omega)$ over all frequencies, thus getting back the total energy density, rediscover the law $u = aT^4$ you have previously derived on thermodynamic grounds. Give an explicit expression for the constant a.
   Hint: $\int_0^\infty \mathrm{d}x\, x^3/(\mathrm{e}^x - 1) = \frac{\pi^4}{15}$.

General Physics

(Note: Parts (a),(b),(c) are not related to each other. Compute all numerical results by hand; orders of magnitude are more important here than the number of significant figures.)

(a) Consider a perfectly reflecting "solar sail" placed in outer space at a distance from the Sun equal to the Earth's orbit radius, and facing the Sun.

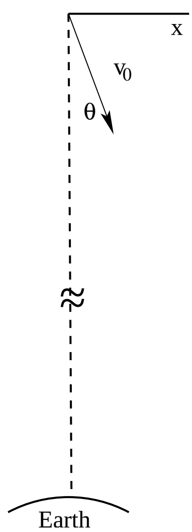
(i) Compute the radiation pressure on that sail due to the Sun's electromagnetic radiation. (The intensity of electromagnetic radiation from the Sun at that distance is 1.4 $\mathrm{kW/m^2}$.)

(ii) If the sail has an area of one square kilometer and is made of aluminum foil of thickness one *micron* (that's $10^{-6}m$), compute its acceleration (in units of g) due to that radiation pressure. (The mass density of aluminum is 2.7×10^3 $\mathrm{kg/m^3}$.)

(iii) At that same distance from the Sun the cloud of particles known as "solar wind" which is continually ejected from the Sun, has an average density of 7 protons per cubic centimeter and moves at an average speed of 400 km/s. Compute the pressure on the sail due to that solar wind, assuming the protons get stuck in the sail (the proton mass is 1.7×10^{-27}kg). Compare the result to that of part (i) above.

(b) The famous supernova SN1987A occurred at a distance of about 5×10^{12} light-seconds from Earth. For the first time in history this supernova was also observed via the burst of neutrinos it emitted. Within that burst, one neutrino of 20 MeV energy and another neutrino of 10 MeV energy arrived on Earth within ten seconds of each other. Assuming that both were emitted simultaneously at the same location within the supernova, derive an approximate upper limit on the neutrino mass, m, in units of eV/c^2. (Hint: You may assume $mc^2 \ll 1$ MeV. Use this to make simplifying approximations early on in yor calculation.)

(c) An astronomical object (for example, a blob of material ejected from a quasar) moves with (actual) speed v_0 at an angle θ off the line of sight from the Earth as shown in the figure at right.



(i) In terms of v_0 and θ find the apparent "sideways" velocity, $v_{x,app}$, of the object as observed from Earth. (You may assume the distance from the Earth to the object to be very much larger than the distance traveled by the object during the period of observation. Do the calculation from the point of view of an observer on Earth.) Check if your result makes sense in the special cases, $\theta = 0$ and $\theta = 90^0$.

(ii) Make a numerical estimate of $v_{x,app}$ if $v_0 \approx 0.99\ c$ and $\theta \approx 6^0$ (which means $\sin\theta \approx 0.1$), and comment on the result.