Carnegie Mellon | Department of Physics | Fall 2010

Graduate Written Qualifying Exam

Day 1 – Classical Physics

Monday, Aug. 16, 2010

***** Please read carefully before beginning *****

On day 1 of this exam you are asked to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

Many of the problems only need a few lines of calculation. If you find yourself in a lengthy calculation, stop and move on. If something appears unclear, don't hesitate to ask.

Good Luck!

Expressions, formula, physical constants, integrals, etc
(which you may find useful although you may not need all of them)
Coordinates: Cartesian (xyz), Cylindrical ($s\phi z$), Spherical ($r\theta\phi$)

$$V = \sum_{n=1}^{\infty} (A_n \sin\alpha_n x + B_n \cos\alpha_n x)\left(C_n e^{\alpha_n y} + D_n e^{-\alpha_n y}\right) + C'$$

$$V = \sum_{n=1}^{\infty} \left(A_n s^n + B_n s^{-n}\right)(C_n \cos n\phi + D_n \sin n\phi) + (F \ln s + G)(H\phi + C')$$

$$V = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-n-1}\right)[C_n P_n(\cos\theta) + D_n Q_n(\cos\theta)] + C'$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} = \frac{\partial f}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z} = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

$$\nabla\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$\begin{aligned}\nabla\times\vec{v} &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z} \\ &= \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{z} \\ &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\ v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\phi}\end{aligned}$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\int_0^\infty dx\ e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad \int_0^\infty dx\ x e^{-ax^2} = \frac{1}{2a} \qquad \int_0^\infty dx\ x^2 e^{-ax^2} = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$\int_0^\infty dx\ x\,e^{-ax} = \frac{1}{a^2} \qquad \int_0^\infty dx\ x^2\,e^{-ax} = \frac{2}{a^3} \qquad \int_0^\infty dx\ x^n\,e^{-ax} = \frac{n!}{a^{n+1}}$$

$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N\,m^2}$ | $\mu_0 = 4\,\pi \times 10^{-7}\ \mathrm{N\,s^2/C^2}$

$c = 1/\sqrt{\epsilon_0\mu_0} = 3.0 \times 10^8\ \mathrm{m/s}$ | $e = 1.602 \times 10^{-19}\ \mathrm{C}$

$h = 6.626 \times 10^{-34}\ \mathrm{J\,s}$ | $\hbar = h/2\pi = 6.582 \times 10^{-22}\ \mathrm{MeV\,s}$

$\hbar c = 197\ \mathrm{MeV\ fm} = 197\ \mathrm{eV\ nm}$ | $(\hbar c)^2 = 0.389\ \mathrm{GeV^2\ mbarn}$

$m_e = 0.511\ \mathrm{MeV}/c^2 \approx 10^{-30}\ \mathrm{kg}$ | $m_{proton} = 938\ \mathrm{MeV}/c^2$

$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$ | $1\ \mathrm{eV}/c^2 = 1.783 \times 10^{-36}\ \mathrm{kg}$

$k_B = 1.381 \times 10^{-23}\ \mathrm{J/K}$ | $k_B = 8.617 \times 10^{-5}\ \mathrm{eV/K}$

Classical Mechanics

A thin, uniform bar of mass M and length $3L/2$ is suspended by a string of length L and negligible mass, as shown below. [Note: the moment of inertia of a thin uniform bar of length l and mass m about the center of mass, perpendicular to its length is $(1/12)ml^2$.]

(a) In terms of the variables θ and ϕ shown in the figure, what is the position and velocity of the center of mass of the bar in the xy-plane? Adopt a coordinate system centered on the top end of the string and let $(x_1,\ y_1)$ be the coordinates of the connection between string and bar. Write the Lagrangian for arbitrary angles θ and ϕ, and show that the Lagrangian appropriate for small oscillations is

$$L = \frac{1}{2}ML^2\left[(\dot{\phi})^2 + \frac{3}{4}(\dot{\theta})^2 + \frac{3}{2}\dot{\phi}\dot{\theta}\right] + MLg\left[(1-\frac{1}{2}\phi^2) + \frac{3}{4}(1-\frac{1}{2}\theta^2)\right].$$

(b) In the limit of small oscillations, find the Euler-Lagrange equations and show that the equations of motion for the angles θ and ϕ are:

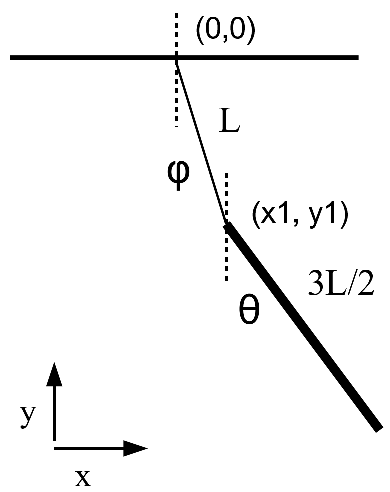
$$L\ddot{\theta} + L\ddot{\phi} + g\theta = 0$$

and

$$L\ddot{\phi} + \frac{3}{4}L\ddot{\theta} + g\phi = 0$$

(c) Again in the limit of small oscillations, write down the form of the normal modes of the system and solve for the frequencies of the normal modes. Describe, both quantitatively and qualitatively, the motion of each normal mode.

(d) Now consider the situation where the system is at rest with $\theta = \phi = 0$. At time t=0 an impulse $F\Delta t$ is applied to the center of the bar in the horizontal direction. Does the impulse excite a single normal mode (which one?) or does it excite a superposition of modes? Briefly explain.



Electricity and Magnetism

("Plasma Physics 101")

A plane electromagnetic (e.m.) wave in vacuum, $\tilde{\mathbf{E}}(z,t) = \hat{\mathbf{y}}\ E_0\ e^{i(kz-\omega t)}$, encounters a lonely free particle of mass m and charge q , initially at rest at the origin. You may assume E_0 to be real.

(a) Neglecting the *magnetic* force, write down the equation of motion of the particle. Given the velocity of the particle as $\tilde{v}(t) = \tilde{v}_0\ e^{-i\omega t}$, solve for the amplitude $\tilde{v}_0$.

(b) Suppose the e.m. wave encounters a dilute plasma of N such free charged particles per unit volume. Show that the conductivity of the plasma is given by $\sigma = i\ N\ q^2/(m\ \omega)$.

(c) Recall that an e.m. wave in any conducting medium must satisfy the "dispersion relation", $\tilde{k}^2 = \mu\ \epsilon\ \omega^2 + i\ \sigma\ \mu\ \omega$ (No need to derive this here.) (i) Use this to find an expression for the minimum frequency (*a.k.a.* "plasma frequency") ω_p for which the wave number $\tilde{k}$ is real. (ii) Make a numerical estimate of ω_p if $N \approx 5 \times 10^{11}/m^3$ which is typical for the (nearly) free electrons in the Earth's ionosphere. What is the implication of the result for the radio communication between NASA and its satellites in outer space?

(d) In the equation of motion in part (a) we neglected the *magnetic* force on the particle. Is that assumption equally good (or bad) at all frequencies? (Hint: compute the ratio of the magnitude of the magnetic force to that of the electric force, and simplify.)

(e) *WOULD A PLASMA SKY BE BLUE?* In other words: suppose the Earth's entire atmosphere was a dilute plasma, what color would we see if we looked straight up at the sky on a sunny afternoon? Do a calculation to find out, and state your reasoning.

Mathematical Physics

A cylinder of radius a and height h is a model for a "quantum pillar", in which a single electron is confined in the interior. The dimensionless time-independent Schrödinger equation is

$$-\nabla^2\psi = E\psi$$

with the Laplace operator in cylindrical coordinates given by

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}.$$

We impose the boundary condition $\psi = 0$ on the surface of the cylinder.

(a) Assuming a solution in the separation of variables form (i.e. a product of three functions), derive differential equations governing each of the three functions. Be sure to fully define every symbol you introduce, and state the boundary conditions applied to each differential equation.

(b) Solve the differential equations related to the angle θ and vertical position z, obtaining the most general solutions consistent with the boundary conditions and any other conditions that the solutions must obey.

(c) The radial equation has two linearly independent solutions. Obtain the limiting forms of both solutions for small r. Only one of these two solutions is applicable to the quantum pillar – state which one, and briefly explain why.

(d) What is the form of the radial equation at large r? Show that in the large r limit the two linearly independent solutions both oscillate. Determine the wavenumber of the oscillation.

(e) What is the most general solution to the radial equation for the quantum pillar that obeys the physical constraints at small r and also the boundary condition at $r = a$?

(f) Assuming E is an allowed energy eigenvalue of the Schrödinger equation, write down the most general wavefunction $\psi(r, \theta, z)$ of the quantum pillar.