Graduate Written Qualifying Exam

Day 2 – Modern Physics

February 10, 2009

***** Please read carefully before beginning *****

On day 2 of this exam you are asked to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

Most of the problems only need a few lines of calculation. If you find yourself in a lengthy calculation, stop and move on. If something appears unclear, don't hesitate to ask.

Good Luck!

Expressions, formula, physical constants, integrals, etc
(which you may find useful although you may not need all of them)

$$Y_{0,0}(\theta,\phi) = \sqrt{\frac{1}{4\pi}} \qquad Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}e^{\pm i\phi}\sin\theta$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2-1) \qquad P_3(x) = \frac{1}{2}(5x^2-3x)$$

$$\oint_C f(z)\,dz = 2\pi i \sum \mathrm{Res}\,[f] \qquad \ln(N!) \approx N\ln(N) - N, \quad \text{for } N \gg 1$$

$$\int_0^\infty dx\; e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad \int_0^\infty dx\; xe^{-ax^2} = \frac{1}{2a} \qquad \int_0^\infty dx\; x^2e^{-ax^2} = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$\int_0^\infty dx\; x\,e^{-ax} = \frac{1}{a^2} \qquad \int_0^\infty dx\; x^2\,e^{-ax} = \frac{2}{a^3} \qquad \int_0^\infty dx\; x^3\,e^{-ax} = \frac{6}{a^4}$$

$\epsilon_0 = 8.854\times10^{-12}\ \mathrm{C^2/N\,m^2}$ | $\mu_0 = 4\,\pi\times10^{-7}\ \mathrm{N\,s^2/C^2}$

$c = 1/\sqrt{\epsilon_0\mu_0} = 3.0\times10^8\ \mathrm{m/s}$ | $e = 1.602\times10^{-19}\ \mathrm{C}$

$h = 6.626\times10^{-34}\ \mathrm{J\,s}$ | $\hbar = h/2\pi = 6.582\times10^{-22}\ \mathrm{MeV\,s}$

$\hbar c = 197\ \mathrm{MeV\ fm} = 197\ \mathrm{eV\ nm}$ | $(\hbar c)^2 = 0.389\ \mathrm{GeV^2\ mbarn}$

$m_e = 0.511\ \mathrm{MeV}/c^2$ | $m_{proton} = 938\ \mathrm{MeV}/c^2$

$m_\mu = 105.7\ \mathrm{MeV}/c^2$ | $m_{\pi^0} = 135\ \mathrm{MeV}/c^2$

$1\ \mathrm{eV} = 1.602\times10^{-19}\ \mathrm{J}$ | $1\ \mathrm{eV}/c^2 = 1.783\times10^{-36}\ \mathrm{kg}$

$k_B = 1.381\times10^{-23}\ \mathrm{J/K}$ | $N_A = 6.022\times10^{23}\ \mathrm{mol^{-1}}$

$1\ \mathrm{Mpc} = 3.086\times10^{22}\ \mathrm{m}$ | $G_N = 6.674\times10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$

IV. Quantum Mechanics

(a) Consider the Schrödinger equation of a charged particle in the xy-plane

$$\frac{1}{2m}(-i\hbar\,\vec{\nabla} - \frac{e}{c}\vec{A})^2\,\psi(x,y) = E\,\psi(x,y).$$

Suppose the particle moves in a constant magnetic field $\vec{B} = B_0\hat{z}$. We may choose a gauge where

$$\vec{A} = x\,B_0\,\hat{y}.$$

Prove that the particle's momentum in the y-direction is conserved.

(b) Assuming that the momentum of the particle in the y-direction is given by $\hbar k_y$ and $\psi(x,y) = \phi(x)e^{ik_y y}$, show that the Schrödinger equation reduces to

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\,\phi + \frac{1}{2}m\omega^2(x - x_c)^2\,\phi = E_x\,\phi$$

and give the values of ω, x_c and E_x. We will refer to x_c as a "center". What familiar equation have you derived?

(c) Suppose that the system is a plane of dimensions $L_x \times L_y$ with periodic boundary conditions in the y-direction: $\psi(x,y) = \psi(x, y+L_y)$. Determine the allowed values of k_y consistent with this boundary condition.

(d) Briefly describe the probability density $P(x,y)$ for a particle in the lowest energy level.

(e) Estimate the degeneracy of the lowest energy level by evaluating the spacing Δx_c between adjacent centers.

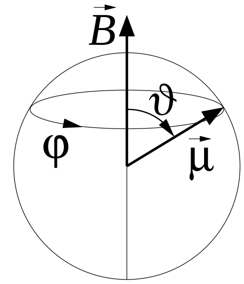
(f) Suppose we wish to add spin to the Hamiltonian given in part (a). This can be accomplished by writing:

$$\frac{1}{2m}[\,(i\hbar\,\vec{\nabla} - \frac{e}{c}\vec{A})\cdot\sigma\,]^2\,\psi(x,t) = E\,\psi(x,t).$$

Assuming this is the correct procedure to add spin, calculate the gyromagnetic ratio g for the electron. Recall, g is defined via $\mu = \frac{ge}{2mc}\,\vec{S}$ and the coupling to the magnetic field is given by $H = -\vec{\mu}\cdot\vec{B}$. Hint: Write $\sigma_i\sigma_j$ in terms of the sum of a commutator and anti-commutator.

V. Statistical Mechanics and Thermodynamics

A magnetic dipole $\vec{\mu}$ is placed into a homogeneous magnetic field $\vec{B}$. The energy (Hamiltonian) of the dipole is $H = -\vec{\mu} \cdot \vec{B}$. The usual spherical polar angles ϑ and φ, that measure the dipole orientation relative to the field direction, are suitable degrees of freedom. We will neglect their conjugate momenta in all that follows.



(a) Write down the classical partition function Z of this system.
Hint: Express $\vec{\mu} \cdot \vec{B}$ in spherical polar coordinates.

(b) Evaluate the integrals obtained above and show that (up to an irrelevant constant prefactor) $Z = \frac{1}{y} \sinh(y)$, where we introduced the convenient dimensionless variable $y := \frac{\mu B}{kT}$.

(c) Let the canonically averaged alignment of the dipole with the field be $\langle \cos(\vartheta) \rangle$. Write this expression *explicitly* as Boltzmann-weighted integrals over the degrees of freedom.

(d) Take your answer from part (c) and show that

$$\langle \cos(\vartheta) \rangle = \frac{1}{Z} \frac{\partial Z}{\partial y} = \frac{\partial \ln(Z)}{\partial y}.$$

(e) Show that the magnetization $M = \mu \langle \cos(\vartheta) \rangle$ is given by $M = -\frac{\partial F}{\partial B}$, where $F(T, B) = -kT \ln(Z)$ is the free energy.

(f) Perform the derivative in part (d) and show that $M = \mu \mathcal{L}(y)$, where we defined the so-called "Langevin function" $\mathcal{L}(x) := \coth(x) - \frac{1}{x}$.

(g) Work out $M(B)$ in the limit of small B. Suggestion: Expand $\mathcal{L}(y)$ for small y by Taylor-expanding all exponential functions entering the *coth*(y) part up to <u>third</u> order in y.

(h) What is the isothermal magnetic susceptibility $\chi_T = \left(\frac{\partial M}{\partial B}\right)_T$ at small B-field?

(i) What is the behavior of $M(B)$ for $B \to \infty$?

(k) Sketch the function $M(B)$ and try to include all results that you know by now. It will be convenient to use $y = \mu B/kT$ as the horizontal axis (or, equivalently, to measure B in units of kT/μ).

VI. General Physics

The Mössbauer effect refers to the resonant and recoil-free emission and absorption of gamma rays from nuclei. Consider the case of Fe^{57}, which has a nuclear ground state with spin and parity $J^P = 1/2^-$ and an excited state with $J^P = 3/2^-$ at 14.4 keV above the ground state. (Do all numerical calculations in this problem by hand to $\sim 20\%$ accuracy.)

(a) The excited state is known to have a half-life of 98 nano-seconds. What is the natural line width of the state expressed in eV?

(b) By how much is the energy of the emitted photon shifted due to the recoil of the nucleus? Again, give your answer in eV.

(c) How much broadening of the gamma-ray transition from the excited state to the ground state is expected if the emitting nucleus is in classical thermal equilibrium at room temperature? Compare your result to the natural line width from part (a).

(d) The Mössbauer effect arises because a "recoilless" fraction of the gamma rays emitted exhibit the natural line width instead of the recoil-shifted and thermally-broadened width. Explain in words the physics principles that make this effect possible to observe.

(e) The Zeeman effect, describing the splitting of spectral lines by an atom placed in a static magnetic field, plays an important role in Mössbauer spectroscopy. Soft iron has a strong magnetic field at the nucleus due to the magnetism of the electrons in the lattice. This field splits the energy levels of the $J^P = 1/2^-$ ground state and the first excited state with $J^P = 3/2^-$.

(i) Draw a schematic energy level diagram showing how the given states are split, labeling the states according to their magnetic quantum numbers m_z. There is no calculation needed for this part. You may ignore quadrupole effects.

(ii) On your diagram show all possible photon absorption transitions from the ground state to the first excited state for the lowest allowed multipole transitions. What type of transition characterizes the main effect?

(f) The energy level splittings depend on the strength of the magnetic moments, μ, of the states. Magnetic moments are measured in units of "magnetons". Use the example of a point particle of mass m and charge e in a circular orbit, to show that a natural definition for a magneton is given by

$$\mu = \frac{e\hbar}{2m}.$$