Graduate Written Qualifying Exam

Day 2 – Modern Physics

August 19, 2009

***** Please read carefully before beginning *****

On day 2 of this exam you are asked to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

Many of the problems only need a few lines of calculation. If you find yourself in a lengthy calculation, stop and move on. If something appears unclear, don't hesitate to ask.

Good Luck!

Expressions, formula, physical constants, integrals, etc
(which you may find useful although you may not need all of them)

$$Y_{0,0}(\theta,\phi)=\sqrt{\frac{1}{4\pi}} \qquad Y_{1,0}(\theta,\phi)=\sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{1,\pm1}(\theta,\phi)=\mp\sqrt{\frac{3}{8\pi}}e^{\pm i\phi}\sin\theta$$

$$P_0(x)=1 \qquad P_1(x)=x \qquad P_2(x)=\frac{1}{2}(3x^2-1) \qquad P_3(x)=\frac{1}{2}(5x^3-3x)$$

$$\oint_C f(z)\,dz = 2\pi i \sum \mathrm{Res}\,[f] \qquad \ln(N!) \approx N\ln(N)-N, \quad \text{for } N\gg 1$$

$$\int_0^\infty dx\ e^{-ax^2}=\frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad \int_0^\infty dx\ xe^{-ax^2}=\frac{1}{2a} \qquad \int_0^\infty dx\ x^2e^{-ax^2}=\frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$\int_0^\infty dx\ x\,e^{-ax}=\frac{1}{a^2} \qquad \int_0^\infty dx\ x^2\,e^{-ax}=\frac{2}{a^3} \qquad \int_0^\infty dx\ x^n\,e^{-ax}=\frac{n!}{a^{n+1}}$$

$\epsilon_0 = 8.854\times10^{-12}\ \mathrm{C^2/N\,m^2}$ | $\mu_0 = 4\,\pi\times10^{-7}\ \mathrm{N\,s^2/C^2}$

$c = 1/\sqrt{\epsilon_0\mu_0} = 3.0\times10^{8}$ m/s | $e = 1.602\times10^{-19}$ C

$Z_0 = \sqrt{\mu_0/\epsilon_0} \sim 376.7\ \Omega$ | $N_A = 6.022\times10^{23}\ \mathrm{mol^{-1}}$

$h = 6.626\times10^{-34}$ J s | $\hbar = h/2\pi = 6.582\times10^{-22}$ MeV s

$\hbar c = 197$ MeV fm $= 197$ eV nm | $(\hbar c)^2 = 0.389\ \mathrm{GeV^2}$ mbarn

$m_e = 0.511\ \mathrm{MeV}/c^2$ | $m_{proton} = 938\ \mathrm{MeV}/c^2$

$m_\mu = 105.7\ \mathrm{MeV}/c^2$ | $m_{\pi^0} = 135\ \mathrm{MeV}/c^2$

1 eV $= 1.602\times10^{-19}$ J | 1 eV$/c^2 = 1.783\times10^{-36}$ kg

$k_B = 1.381\times10^{-23}$ J/K | $k_B = 8.617\times10^{-5}$ eV/K

1 Mpc $= 3.086\times10^{22}$ m | $G_N = 6.674\times10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$

IV. Quantum Mechanics

This problem concerns the Stark Effect in hydrogen, in which atomic energy levels shift in response to an applied electric field $\mathcal{E}$. In a weak electric field, the ground state ($n = 1$) energy E varies as

$$\Delta E = -\frac{\alpha}{2}\,\mathcal{E}^2$$

with higher order corrections in the field strength. Note the absence of a first order energy shift proportional to $\mathcal{E}$.

(a) Give a symmetry-based explanation (in words, no calculation required) why there is no first order energy shift.

(b) In view of the vanishing first order shift discussed above, briefly explain (in words, no calculation required):

(i) Why does the energy shift to first order in a magnetic field $\mathcal{B}$ in the Zeeman effect?

(ii) Why does the energy shift to first order in an electric field $\mathcal{E}$ in the Stark effect for excited states ($n > 1$)?

(c) Write a formal expression for the second order energy shift ΔE of the ground state due to the Stark Effect. Express your answer in terms of matrix elements of the electric potential V and zero field energy levels E_n. Is ΔE positive or negative? Explain why!

(d) For the hydrogen atom the ground state wave function is

$$|100\rangle = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

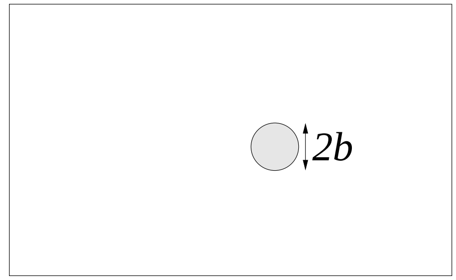
with energy $E_1 = -e^2/2a$. Let the electric field point in the z-direction so that $V = -e\mathcal{E}z$. A strict lower bound (and close estimate) for α can be obtained from your result in part (c) by the following trick: replace all excited state energies E_n with 0, then using the fact that

$$\sum_{nlm} |nlm\rangle\langle nlm| = 1,$$

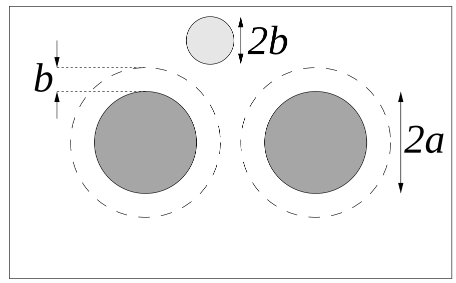
together with the value of the matrix element $\langle 100|z^2|100\rangle$, which you should calculate, discover that you can evaluate ΔE and thus estimate α.

V. Statistical Mechanics and Thermodynamics

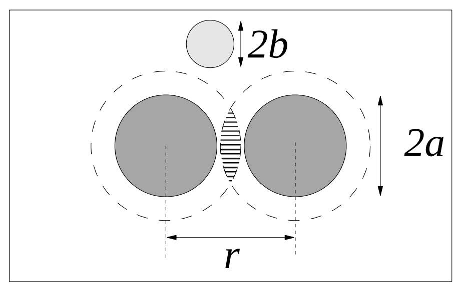
(a) Consider a *three-dimensional* hard-walled box of side-lengths L_x, L_y, and L_z. It encloses a single spherical particle with radius b and mass m. What is the total volume V_0 available to the center of mass of that particle? If the system is at temperature T, determine the canonical partition function $Z_0(1)$ and the Helmholtz free energy $F_0(1)$ of that particle. Express your answer in terms of the thermal de Broglie wavelength $\lambda_{\mathrm{dB}} = h/\sqrt{2\pi m k_{\mathrm{B}} T}$. Ignore gravity.



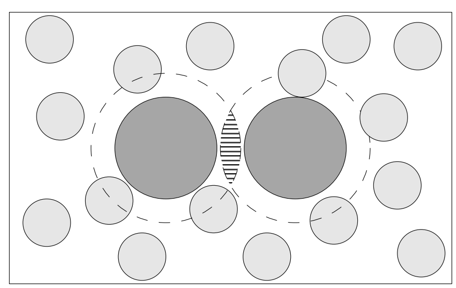
(b) We now add two bigger spherical particles of radius $a \gg b$ at *fixed* positions sufficiently far away from the walls and each other. None of the three particles can overlap and there is no interaction between them. What is now the total volume V_2 available to the small particle? What is its partition function $Z_2(1)$ and Helmholtz free energy $F_2(1)$?



(c) Let the two bigger particles approach each other to a center-to-center distance r smaller than $2(a+b)$ but bigger than $2a$. Explain qualitatively how the available volume for the small particle changes from V_2 to $V_{2'}$, for instance by referring to the shaded lens-shaped region of volume V_{lens} in the figure to the right. Re-express the partition function $Z_{2'}(1)$ and the Helmholtz free energy $F_{2'}(1)$ of the small particle?



(d) We next put $N \gg 1$ small particles into the box from part (c). Assume that their density is small enough such that we can treat them as an *ideal gas*, i.e., we can neglect their interactions with each other. Under this simplified assumption, and given that we know the partition function $Z_{2'}(1)$, what are the partition function $Z_{2'}(N)$ and the Helmholtz free energy $F_{2'}(N)$ of the system of N small particles in the presence of the two bigger (fixed) spheres?



(e) $F_{2'}(N)$ depends on the volume V_{lens} of the lens-shaped region. Since we can safely assume $V_{\mathrm{lens}} \ll V_2$, expand $F_{2'}(N)$ up to *first order* in V_{lens}/V_2.

(f) Your answer to part (e) shows that when the two *big spheres* come sufficiently close together, the free energy of the system of *small spheres* decreases by some amount. Using the fact that the small spheres were treated in ideal gas approximation, simplify this reduction term and discuss it in terms of work.

(g) Since V_{lens} depends on the distance r between the two big particles, the reduction term can be viewed as a small-sphere-induced *attractive interaction potential* $U(r)$ between the bigger spheres. Such an effect is common in colloidal science and referred to as "depletion attraction". With this in mind, calculate and sketch $U(r)$ for all values of r. [Hint: You may use the fact that a spherical cap of height h cut off a sphere of radius R has a volume $V_{\text{cap}} = \pi h^2(R - \frac{1}{3}h)$.]

(h) Explain how the same effect that leads to attractions between big spheres can also lead to an attraction of a single big sphere to one of the container walls.

VI. General Physics

(a) Consider a classical point charge q moving in a uniform static magnetic field $\vec{B}$ pointing in the z-direction. Discuss the motion of this point charge using relevant equations. What are the classical constants of motion?

(b) Consider a sphere with uniform volume charge and mass densities spinning at angular frequency ω about an axis $\hat{z}$. The sphere has radius R, total charge Q and mass M. Using SI units, work out the classical magnetic moment of the spinning sphere in terms of its angular momentum. [Hint: classically the magnetic moment $\vec{m}$ of a current loop of area A and current I is $\vec{m} = \hat{z}IA$.]

(c) For a quantum mechanical but spinless electron in a uniform magnetic field in z-direction, the energy levels are $E_n = \hbar\omega_c(n + 1/2)$ with a classical cyclotron frequency $\omega_c = eB/m$. This result neglects the free-particle component of the electron's motion along $\hat{z}$.

(i) Draw and label with quantum number n a diagram showing the first four energy levels for the quantum mechanical but spinless electron in a constant magnetic field.

(ii) Determine a numerical value for the level separation ΔE in a 1 T magnetic field.

(d) If the electron is in equilibrium at temperature $T = 0.03$ K, what is the probability that any level other than the ground state will be occupied for the spinless electron in the 1 T magnetic field? If the temperature is 300 K, estimate the average value of n for the electron.

(e) The detailed quantum physics of the magnetic moment of the electron is contained in a g-factor g_e that multiplies the classical gyromagnetic ratio. With spin, there is an additional magnetic interaction that will shift the energy levels.

(i) Express the energy of the electron in terms of n and the spin projection quantum number m_s, taking the quantization axis as the $\vec{B}$ direction.

(ii) Show the relation of the new states to the zero-spin states from (c), assuming that g_e is exactly 2. Draw and label with (n, m_s) a new energy level diagram to the right of your diagram from part (c).

(iii) Focusing on the $n = 0$ and $n = 1$ energy levels, draw another energy level diagram for the case if g_e is slightly larger than 2.0.

(f) The g-factor for the electron has been measured with a precision better than one part in 10^{12}. This is an incredible success of experimental science. Assuming that you do not know how this experiment was carried out, we would like you to invent an apparatus to measure the g-factor of the electron, based on your previous analysis. Comment qualitatively on what the experimental apparatus must include. What signal will you measure that depends on g? What properties of your method will limit its accuracy?