Department of Physics

Graduate Written Qualifying Exam Day 1 – Classical Physics

August 18, 2009

*** Please read carefully before beginning ***

On day 1 of this exam you are asked to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

Many of the problems only need a few lines of calculation. If you find yourself in a lengthy calculation, stop and move on. If something appears unclear, don't hesitate to ask.

Good Luck!

Expressions, formula, physical constants, integrals, etc (which you may find useful although you may not need all of them)

$$\begin{split} Y_{0,0}(\theta,\phi) &= \sqrt{\frac{1}{4\pi}} & Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta & Y_{1,\pm 1}(\theta,\phi) = \mp \sqrt{\frac{3}{8\pi}}e^{\pm i\phi}\sin\theta \\ P_0(x) &= 1 & P_1(x) = x & P_2(x) = \frac{1}{2}(3x^2 - 1) & P_3(x) = \frac{1}{2}(5x^3 - 3x) \\ \int_C^\infty f(z)\,dz &= 2\pi i\sum\operatorname{Res}\left[f\right] & \ln\left(N!\right) \approx N\,\ln(N) - N, \quad \text{for } N \gg 1 \\ \int_0^\infty dx\,\,e^{-ax^2} &= \frac{1}{2}\sqrt{\frac{\pi}{a}} & \int_0^\infty dx\,\,xe^{-ax^2} = \frac{1}{2a} & \int_0^\infty dx\,\,x^2e^{-ax^2} = \frac{1}{4}\sqrt{\frac{\pi}{a^3}} \\ \int_0^\infty dx\,\,xe^{-ax} &= \frac{1}{a^2} & \int_0^\infty dx\,\,x^2\,e^{-ax} = \frac{2}{a^3} & \int_0^\infty dx\,\,x^n\,e^{-ax} = \frac{n!}{a^{n+1}} \\ \epsilon_0 &= 8.854 \times 10^{-12}\,\operatorname{C}^2/\mathrm{N}\,\mathrm{m}^2 & \mu_0 = 4\,\pi \times 10^{-7}\,\mathrm{N}\,\mathrm{s}^2/\mathrm{C}^2 \\ c &= 1/\sqrt{\epsilon_0\mu_0} = 3.0 \times 10^8\,\mathrm{m/s} & e = 1.602 \times 10^{-19}\,\mathrm{C} \\ Z_0 &= \sqrt{\mu_0/\epsilon_0} \sim 376.7\,\Omega & N_A = 6.022 \times 10^{23}\,\mathrm{mol}^{-1} \\ h &= 6.626 \times 10^{-34}\,\mathrm{J}\,\mathrm{s} & h = h/2\pi = 6.582 \times 10^{-22}\,\mathrm{MeV}\,\mathrm{s} \\ \hbar c &= 197\,\,\mathrm{MeV}\,\mathrm{fm} = 197\,\,\mathrm{eV}\,\mathrm{nm} & (\hbar c)^2 &= 0.389\,\,\mathrm{GeV}^2\,\,\mathrm{mbarn} \\ m_e &= 0.511\,\,\mathrm{MeV}/c^2 & m_{\pi^0} &= 135\,\,\mathrm{MeV}/c^2 \\ 1\,\,\mathrm{eV} = 1.602 \times 10^{-19}\,\,\mathrm{J} & 1\,\,\mathrm{eV}/c^2 &= 1.783 \times 10^{-36}\,\mathrm{kg} \\ k_B &= 1.381 \times 10^{-23}\,\,\mathrm{J/K} & k_B &= 8.617 \times 10^{-11}\,\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2} \end{split}$$

I. Classical Mechanics

A wire has been bent into the shape of a curve with $y(x) = b (x/a)^{\lambda}$ with $\lambda > 0$. The wire is rotated about the y-axis with constant frequency ω (see diagram to the right). A bead is able to move along the wire without friction. The gravitational acceleration (magnitude g) is straight downwards (negative y-direction) in the diagram.



(a) Write down the speed of the bead in terms of components along the curve (i.e. in the x-y plane) and perpendicular to it.

(b) Making use of your result from part (a), compute the kinetic energy of the system. Write down the Lagrangian, in terms of \dot{x} , \ddot{x} , y' and y'', where $\dot{x} = \frac{dx}{dt}$ and $y' = \frac{dy}{dx}$.

(c) Compute Lagrange's equations of motion and use them to show that the equilibrium value x_0 of the bead's x-coordinate is given by

$$x_0 = a \left(\frac{a^2 \omega^2}{\lambda g b}\right)^{1/(\lambda-2)}$$
 for $\lambda \neq 2$.

(d) Check your answer to part (c) by using a force balance approach.

(e) Perturb the bead's position from x_0 by a small value (i.e. $x = x_0 + \delta$) and show that to first order in δ ,

$$\ddot{\delta}(1+y'(x_0)^2) = \delta[\omega^2 - g \, y''(x_0)]$$

(f) Find the frequency Ω of small oscillations about x_0 . For what values of λ is there oscillatory motion about the equilibrium point?

(g) What is unique about the situation when $\lambda = 2$ and the bead is in equilibrium on the curve?

II. Electromagnetism

A parallel plate capacitor consisting of two circular plates each of radius r_0 is centered at the z-axis with the plates at $z = \pm d/2$ parallel to the xy-plane. A time dependent potential $\phi(t) = V_0 \cos(\omega t)$ is supplied to the plates by suitable wires. Assume $d \ll r_0$ so that fringing effects can be ignored.

(a) Determine the electric field \vec{E} and magnetic field \vec{B} between the plates of the capacitor as a function of time.

(b) Calculate the Poynting vector \vec{S} between the plates of the capacitor.

(c) Calculate the total charge Q(t) on the plates of the capacitor and express the current I(t) passed onto the plates as a function of initial charge $Q_0 = Q(t = 0)$.

(d) Suppose you want to measure the magnitude of the magnetic field $|\vec{B}|$ at a point P between the plates of the capacitor using a piece of wire and an oscilloscope. Describe your experimental setup (with a sketch) and explain how you estimate $|\vec{B}|$.

(e) From a far distance, the capacitor can be viewed as a radiating dipole which emits a total power of

$$\langle P \rangle = \frac{Z_0 \, Q_0^2 \, d^2 \, \omega^4}{12 \pi \, c^2},$$

where Z_0 is the impedance of free space. Determine the resistance of the wire joining the capacitor that would give the same average ohmic power loss (to heat) as the oscillating dipole emits in form of radiation. Express your answer in ohms as a function of d and λ , the wave length of the radiation.

(f) For this final part, we consider a rectangular parallel plate capacitor with the plates having lengths x_0 and y_0 again parallel to the xy-plane. Surface charges $\pm \sigma_0$ provide a static electric field \vec{E}_0 between the plates. Determine the magnitude of the electric field seen by an observer moving with relativistic velocity v_0 along the x-axis. What is the electric field if the observer moves with v_0 along the z-axis?

III. Mathematical Physics

Let \mathcal{P}_n be the collection of all polynomials of the form

$$P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n,$$

where the a_i are complex numbers, which may be either zero or nonzero.

(a) Argue that for a fixed n, \mathcal{P}_n is a (complex) linear vector space satisfying the usual rules for such spaces.

(b) Show that the derivative operator D = d/dz that maps \mathcal{P}_n into itself is a *linear* operator.

(c) Indicate a collection of polynomials that form a basis of the space \mathcal{P}_n . What is the dimension of this space? What are the matrix elements of the derivative operator D relative to the basis you defined? [Hint: This is relatively straightforward if you choose the right basis.]

(d) Show that P(z) is determined everywhere by its values on the unit circle |z| = 1 by writing down an explicit formula for the coefficients a_j in terms of these values. Be sure and give some indication why your formula is correct or how you derived it.

(e) Consider the (n+1)-component vector $\vec{v} = (v_0(z), v_1(z), \dots, v_n(z))$, where $v_j = D^j P$; that means, $v_0(z)$ is $P(z), v_1(z)$ is the first derivative of P(z), and so forth. Show that if Pis in \mathcal{P}_n , the column vector \vec{v} satisfies a linear first-order differential equation of the form

$$\frac{d\vec{v}}{dz} = M\vec{v},\tag{1}$$

where M is an $(n + 1) \times (n + 1)$ matrix with constant coefficients, whose form you should determine.

(f) As well as polynomial solutions in \mathcal{P}_n , does the differential Eq. (1) in part (e) have any exponential solutions $e^{\alpha z}$, or perhaps a polynomial in z times $e^{\alpha z}$, with α being some nonzero complex number? Give reasons for your answer.