

Graduate Written Qualifying Exam

Day 2 – Modern Physics

August 20, 2008

***** Please read carefully before beginning *****

On day 2 of this exam you are asked to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

Most of the problems only need a few lines of calculation. If you find yourself in a lengthy calculation, stop and move on. If something appears unclear, don't hesitate to ask.

Good Luck!

Expressions, formula, physical constants, integrals, etc

(which you may find useful although you may not need all of them)

$$Y_{0,0}(\theta, \phi) = \sqrt{\frac{1}{4\pi}} \quad Y_{1,0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta \quad Y_{1,\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} e^{\pm i\phi} \sin \theta$$

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{1}{2}(3x^2 - 1) \quad P_3(x) = \frac{1}{2}(5x^2 - 3x)$$

$$\oint_C f(z) dz = 2\pi i \sum \text{Res}[f] \quad \ln(N!) \approx N \ln(N) - N, \quad \text{for } N \gg 1$$

$$\int_0^\infty dx e^{-ax^2} = \frac{1}{2} \sqrt{\frac{\pi}{a}} \quad \int_0^\infty dx x e^{-ax^2} = \frac{1}{2a} \quad \int_0^\infty dx x^2 e^{-ax^2} = \frac{1}{4} \sqrt{\frac{\pi}{a^3}}$$

$$\int_0^\infty dx x e^{-ax} = \frac{1}{a^2} \quad \int_0^\infty dx x^2 e^{-ax} = \frac{2}{a^3} \quad \int_0^\infty dx x^3 e^{-ax} = \frac{6}{a^4}$$

$$\epsilon_0 = 8.854 \times 10^{-12} \text{ C}^2/\text{N m}^2 \quad \mu_0 = 4\pi \times 10^{-7} \text{ N s}^2/\text{C}^2$$

$$c = 1/\sqrt{\epsilon_0 \mu_0} = 3.0 \times 10^8 \text{ m/s} \quad e = 1.602 \times 10^{-19} \text{ C}$$

$$h = 6.626 \times 10^{-34} \text{ J s} \quad \hbar = h/2\pi = 6.582 \times 10^{-22} \text{ MeV s}$$

$$\hbar c = 197 \text{ MeV fm} = 197 \text{ eV nm} \quad (\hbar c)^2 = 0.389 \text{ GeV}^2 \text{ mbarn}$$

$$m_e = 0.511 \text{ MeV}/c^2 \quad m_{proton} = 938 \text{ MeV}/c^2$$

$$m_\mu = 105.7 \text{ MeV}/c^2 \quad m_{\pi^0} = 135 \text{ MeV}/c^2$$

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J} \quad 1 \text{ eV}/c^2 = 1.783 \times 10^{-36} \text{ kg}$$

$$k_B = 1.381 \times 10^{-23} \text{ J/K} \quad N_A = 6.022 \times 10^{23} \text{ mol}^{-1}$$

$$1 \text{ Mpc} = 3.086 \times 10^{22} \text{ m} \quad G_N = 6.674 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$$

IV. Quantum Mechanics

The alkali atoms H, Li, Na, K, Rb, Cs and Fr can be labeled by an integer $n = 1, 2, 3, 4, 5, 6$ and 7, respectively. The electronic structure of the n^{th} alkali atom can be approximated by a single valence electron in the nS state of a Coulomb field generated by the closed shells of the inner electrons and the nucleus.

- (a) Estimate the first ionization energy of Rb.
- (b) An alkali atom in its ground state has two contributions to its spin: The electronic spin $\vec{S}$ with quantum number $s = 1/2$ and its nuclear spin $\vec{I}$ with quantum number i . Note that the nuclear spin does not need to be spin $1/2$. How many total spin states are possible?
- (c) Defining the total spin $\vec{F} = \vec{S} + \vec{I}$, write down two sets of complete sets of commuting observables associated with spin. One set should include F^2 , the other one should not.
- (d) The Hamiltonian for a single atom includes a hyperfine term that can be written as

$$H = \frac{2E_{hf}}{(2i+1)\hbar^2} \vec{I} \cdot \vec{S}$$

This interaction splits the energy levels into two multiplets. Assuming that $i = 3/2$ write down the quantum numbers of these multiplets along with their corresponding eigenvalues of the hyperfine Hamiltonian.

- (e) Suppose the nuclear spin is $i = 1/2$. Write down the spin wave function for the energy eigenstates in the S_z, I_z basis. Use the notation $|i_z; s_z\rangle$, so that, for instance, a state with the electron and nuclear spin being $1/2$ is written as $|1/2; 1/2\rangle$. Write down the energy eigenvalues for each state.

- (f) Suppose we now place the atom in an external magnetic field $\vec{B}$. The magnetic moment μ of the atom is dominated by the term proportional to the electron spin. The magnetic term in the Hamiltonian is

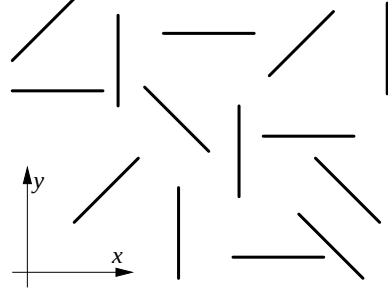
$$H_{mag} = -2\frac{\mu}{\hbar} \vec{S} \cdot \vec{B}$$

Assuming that $\mu B \ll E_{hf}$, calculate the approximate shift in the energy levels. Are there any degenerate states left? If so, label them. Note that you may ignore the effects of the nuclear spin since the nuclear magnetic moment is suppressed.

- (g) Why is the nuclear magnetic moment suppressed?
- (h) Prove that the full Hamiltonian, when acting on a state, preserves the total angular momentum in the z -direction. Using this fact find two states with fixed i_z and s_z (i.e. of the form $|i_z, s_z\rangle$) which are eigenstates of the full Hamiltonian. These states play a special role in atomic traps.

V. Statistical Mechanics and Thermodynamics

Consider a *two-dimensional* gas of N rodlike particles in the xy -plane, all having mass m , length ℓ , and thickness zero (“line-particles”). We simplify the rotational degree of freedom by assuming that *the rods can only assume M different orientations*. This also eliminates the need to consider rotational energy.



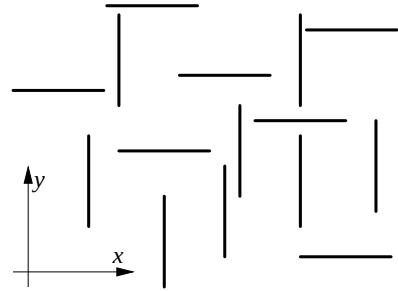
(a) In a first step we *neglect any particle interactions*. You can think of this system as a *mixture of M ideal gases*, each one consisting of N_i particles in orientation i . We define the total particle number $N = \sum_{i=1}^M N_i$, the fractions $n_i = N_i/N$ of particles in orientation i , the overall density $\rho = N/A$, where A is the area of the region in which all particles move, and the thermal de Broglie wavelength $\lambda = h/\sqrt{2\pi m k_B T}$.

Give a detailed derivation showing that the free energy per particle, f_{ideal} , in the limit that all $N_i \gg 1$, is given by

$$\beta f_{\text{ideal}} \equiv \frac{F_{\text{ideal}}(\{N_i\}, A, T)}{N k_B T} = [\ln(\rho \lambda^2) - 1] + \sum_{i=1}^M n_i \ln n_i.$$

(b) The expression in square brackets is the free energy of an ideal gas of N *structureless* particles. What is the physical meaning of the additional term $\sum_i n_i \ln n_i$?

(c) From now on we simplify the situation further by assuming that there are only $M = 2$ different orientations: one along the x -direction, and one along the y -direction. Let n_x and n_y be the fractions of rods pointing along the x - and y -direction, respectively. Show that the ideal free energy is minimized at $n_x = \frac{1}{2}$, *i.e.*, for an *isotropic* distribution of rods.



(d) If the density increases, interactions between the rods become important. Specifically, the *particles cannot overlap*. Within a second order virial expansion this adds the following “collision term” to the free energy:

$$\beta f_{\text{collision}} = \frac{F_{\text{collision}}}{N k_B T} = \frac{1}{2} \rho \sum_{i,j \in \{x,y\}} n_i n_j A_{ij}, \quad (1)$$

where A_{ij} is the *excluded area* between two rods of orientation i and j , *i.e.*, the area of the region which the center of mass of one of the rods cannot occupy due to the presence of the other. Determine all four A_{ij} . A sketch might help.

(e) Show that $\beta f_{\text{collision}} = \rho \ell^2 (\frac{1}{4} - \varepsilon^2)$, where we introduced $\varepsilon = n_x - \frac{1}{2}$ as an “*order parameter*” that measures the deviation of the system from the isotropic state.

(f) Since f_{ideal} is minimized at $n_x = \frac{1}{2}$, we expand it for small ε around this point. By considering $f_{\text{total}} = f_{\text{ideal}} + f_{\text{collision}}$ and using the approximation

$$\beta f_{\text{ideal}} = \text{const.} + 2\varepsilon^2 + \frac{4}{3}\varepsilon^4 + \mathcal{O}(\varepsilon^6),$$

show that there is a *critical density* ρ_c such that for $\rho \leq \rho_c$ the system is isotropic, while for $\rho > \rho_c$ the system prefers a “nematic phase”, in which the value of the order parameter ε that minimizes f_{total} is either bigger or smaller than the isotropic value $\varepsilon = 0$. What happens physically to the system when it enters the “nematic phase”?

VI. General Physics

A recent demonstration of wave-particle duality of a large molecule has been demonstrated with C_{60} molecules (O. Nairz, M. Arndt, and A. Zeilinger, *Amer. J. Phys.* **71** (2003) 319). In this experiment interference was observed in a matter version of Young's famous double slit experiment.

(a) Sketch a diagram of Young's classic double slit experiment, which was originally performed with light, and label each element. Derive the condition to observe constructive interference in the experiment.

(b) Some issues encountered in a single particle interferometry experiment of a large molecule like C_{60} , that are not encountered in the same experiment with light, are a velocity distribution in the beam of molecules and the small spacing of the interference orders that requires ionization of the molecules for detection of the interference patterns.

(i) *With these specific issues in mind*, draw a block diagram of an apparatus that you might use to observe double slit diffraction with C_{60} molecules. Label each element and briefly describe its function.

(ii) Comment on at least one other way in which this experiment is different (*in a measurable way*) from an optical interference experiment.

(c) What is the angular separation between interference maxima in the observation plane for a beam of molecules with a velocity of 200 m/s and a slit separation of 100 nm given that $m_{C_{60}} = 1.2 \times 10^{-24}$ kg?

(d) In order to observe the interference pattern, the experimentalists had to worry about the width of both the longitudinal velocity distribution as well as the transverse distribution. Discuss how each of these distributions affects the measurement and how you could limit them. Be quantitative in your discussion.