Graduate Qualifying Exam 2005

DAY TWO – Modern Physics

August 23, 2005

Please read carefully before starting –

On this second day of the exam we ask you to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

Physical constants (which you may find useful although you may not need all of them)

$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N\,m^2}$	$\mu_0 = 4\,\pi \times 10^{-7}\ \mathrm{N\,s^2/C^2}$
$c = 1/\sqrt{\epsilon_0 \mu_0} = 3.0 \times 10^8\ \mathrm{m/s}$	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
$h = 6.626 \times 10^{-34}\ \mathrm{J\,s}$	$\hbar = h/2\pi = 6.582 \times 10^{-22}\ \mathrm{MeV\,s}$
$\hbar c = 200\ \mathrm{MeV\,fm}$	$(\hbar c)^2 = 0.389\ \mathrm{GeV^2\,mbarn}$
$m_e = 0.511\ \mathrm{MeV}/c^2$	$m_p = 938.3\ \mathrm{MeV}/c^2$
$m_\mu = 105.7\ \mathrm{MeV}/c^2$	$m_{\pi^0} = 135\ \mathrm{MeV}/c^2$
$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$	$1\ \mathrm{eV}/c^2 = 1.783 \times 10^{-36}\ \mathrm{kg}$
$k_B = 1.381 \times 10^{-23}\ \mathrm{J/K}$	$N_A = 6.022 \times 10^{23}\ \mathrm{mol^{-1}}$
$1\ \mathrm{Mpc} = 3.086 \times 10^{22}\ \mathrm{m}$	$G_N = 6.674 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$

IV. Quantum Mechanics

Consider a particle of mass m moving in a one-dimensional potential with infinitely high walls placed at $x = \pm a$. The potential vanishes between the walls.

(a) Write down the eigenenergies and the associated eigenstates of the system in the x basis.

(b) Suppose that at $t = 0$ the particle is prepared in a state which can be well approximated by the wave function

$$\psi(x) = N e^{-x^2/\delta^2} ,$$

where $\delta \ll a$.
(i) Determine N and then determine the wave function at some later time t_0.
(ii) What is the probability of finding the particle in the m-th energy eigenstate at some later time t_0 ?

(c) Suppose that the width of the well is now taken to zero in such a way that the potential can be written as

$$V(x) = - \mid V_0 \mid \delta(x) ,$$

where V_0 is a constant. Does this system have a bound state? If so, what is the wave function and energy eigenvalue? If not, explain why not.

(d) Suppose now that we generalize the potential to look like

$$V(x) = - \mid V_0 \mid \delta(x-a) - \mid V_0 \mid \delta(x+b) .$$

Consider the case $a = b$, what operator commutes with the Hamiltonian that would not if $a \neq b$?

(e) Draw the wave function in the x representation of the lowest energy eigenstate of the system. Do you expect the particle to be more strongly bound in the double well case or in the single well case? Explain your answer.

(f) Now consider the case $a \neq b$. Does the number of operators which commute with the Hamiltonian change?

V. Statistical Mechanics

The following questions will consider a model material with a density of states given by

$$D(\varepsilon) = \begin{cases} X & 0 < \varepsilon < A \\ 0 & A \leq \varepsilon \leq B \\ Y & B < \varepsilon \end{cases}$$

The total number of particles in the system is given by $N = XA$ for both the Fermi-Dirac and Bose-Einstein parts of the problem.

Fermi Statistics

(a) Write the equations for the total energy and number of particles for fermions.

(b) What is the Fermi energy in the limit $T \to 0$?

(c) Find the chemical potential for low (but not zero) temperature.

Bose statistics

(d) Write the equations for the total energy and number of particles for bosons with the given density of states.

(e) Describe qualitatively what is meant by Bose-Einstein condensation.

(f) Calculate the temperature at which the Bose-Einstein condensation takes place for the given density of states.

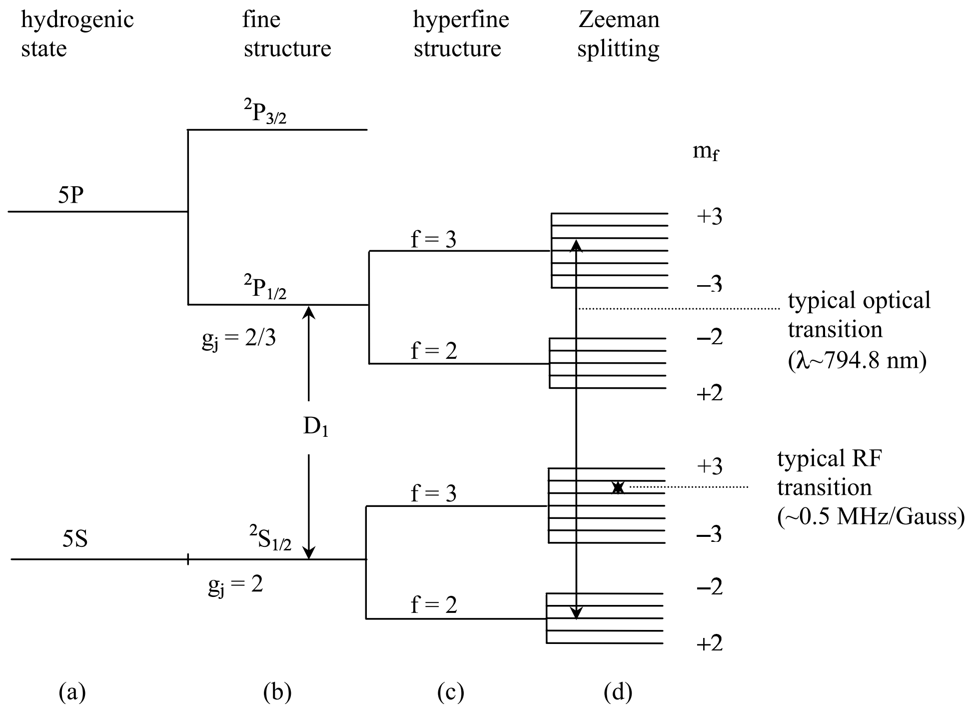

Figure 1: Energy levels (not to scale) of ^{85}Rb as successively weaker interactions are turned on (from a to d).

VI. General Physics

^{85}Rb is an alkali atom, in which there is only one electron outside the closed shells. The figure above shows a schematic energy level diagram for the ground and lowest excited states of ^{85}Rb in a weak magnetic field, showing how the ground and first excited electronic states can be imagined to split into successive sublevels as the various interactions are "turned on". The energy scale is grossly distorted in order to display the hierarchy of structure.

(1) Describe in words what is responsible for each successive splitting from (a) to (d). Write down the form of the corresponding interaction Hamiltonian, taking care to define all symbols.

(2) With reference to the Figure, estimate the energy differences, in eV, between each of the following pairs of states of ^{85}Rb (do all numerical calculations in this problem by hand, to an accuracy of about a factor of two):

(i) $5^2S_{1/2} - 5^2P_{1/2}$
(ii) $5^2P_{1/2} - 5^2P_{3/2}$
(iii) $5^2S_{1/2}(f = 2) - 5^2S_{1/2}(f = 3)$
(iv) $5^2S_{1/2}(f = 2,\ m_f = -1) - 5^2S_{1/2}(f = 2,\ m_f = 0)$ in a magnetic field of 1 $Gauss$.

(3) In thermal equilibrium at 320 K, how many atoms in a mole of rubidium would one expect to find in the $5^2P_{1/2}$ state?

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(*Problem VI, continued*)

(4) What is the difference in the population of the lowest and highest magnetic substate of the ground state in a field of 1 Gauss at a temperature of 320 K?

(5) The optical transitions are known to have a mean lifetime of about 10^{-8} seconds. What is the natural line width of such states, in eV? Show whether this width large or small compared with the broadening of this transition expected if the emitting atoms are in classical thermal equilibrium at 320 K.

(6) Consider the effect of illuminating a cell of ^{85}Rb vapor with left-circularly polarized light which propagates parallel to an external 1 *Gauss* magnetic field. The incident photons are in a narrow range of energies such that they can induce $5^2S_{1/2} \rightarrow 5^2P_{1/2}$ dipole transitions. What will be the population equilibrium among the states of the atoms? You can neglect collision-induced transitions between the magnetic substates.

(7) What is the spin of the nucleus in this atom? What fractional contributions does the nucleus make to (i) the total angular momentum and (ii) the total magnetic moment of a rubidium atom?