Graduate Qualifying Exam 2005

DAY ONE – Classical Physics

August 22, 2005

Please read carefully before starting –

On this first day of this exam you are asked to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

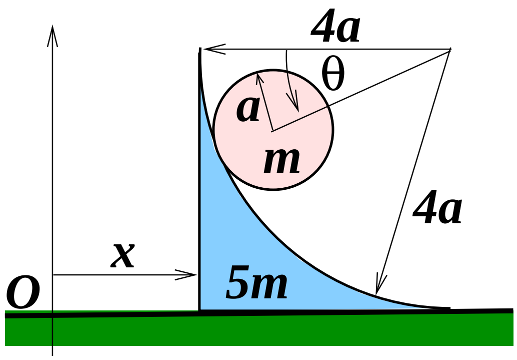
Physical constants (which you may find useful although you may not need all of them)

$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N\,m^2}$	$\mu_0 = 4\,\pi \times 10^{-7}\ \mathrm{N\,s^2/C^2}$
$c = 1/\sqrt{\epsilon_0\mu_0} = 3.0 \times 10^{8}\ \mathrm{m/s}$	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
$h = 6.626 \times 10^{-34}\ \mathrm{J\,s}$	$\hbar = h/2\pi = 6.582 \times 10^{-22}\ \mathrm{MeV\,s}$
$\hbar c = 200\ \mathrm{MeV\,fm}$	$(\hbar c)^2 = 0.389\ \mathrm{GeV^2\,mbarn}$
$m_e = 0.511\ \mathrm{MeV}/c^2$	$m_p = 938.3\ \mathrm{MeV}/c^2$
$m_\mu = 105.7\ \mathrm{MeV}/c^2$	$m_{\pi^0} = 135\ \mathrm{MeV}/c^2$
$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$	$1\ \mathrm{eV}/c^2 = 1.783 \times 10^{-36}\ \mathrm{kg}$
$k_B = 1.381 \times 10^{-23}\ \mathrm{J/K}$	$N_A = 6.022 \times 10^{23}\ \mathrm{mol^{-1}}$
$1\ \mathrm{Mpc} = 3.086 \times 10^{22}\ \mathrm{m}$	$G_N = 6.674 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$

I. Mechanics

A spherical ball of radius a and mass m which is *not* uniformly distributed throughout its volume rolls without slipping down the circular surface of radius of curvature $4a$ of a wedge of mass $5m$ which is free to slide frictionlessly across a smooth horizontal surface, as shown. The density $\rho(r)$ of the ball varies only with the distance r from its center according to

$$\frac{\rho(r)}{\rho(0)} = 1 - \frac{4r}{5a} \ .$$



(a) Show that the moment of inertia of the ball about an axis coming out of the page through the center of the ball is $I = \frac{1}{3}ma^2$.

(b) Let x denote the rightward horizontal displacement of the left vertical edge of the wedge from the origin O, and let θ denote the angle which a line connecting the center of curvature of the surface to the center of the ball makes with the horizontal. Determine the Lagrangian L of the system in terms of x, θ, $\dot{x}$, and $\dot{\theta}$.

(c) From the equations of motion, find expressions for the accelerations $\ddot{x}$ and $\ddot{\theta}$ in terms of x, θ, $\dot{x}$, and $\dot{\theta}$.

(d) If both the ball and the wedge start initially at rest with the ball positioned such that $\theta = 0$, eventually the wedge ends up moving with uniform speed v_f to the left. Determine an expression for v_f, simplified as much as possible.

II. Electromagnetic Radiation and Waves

(a) A 10^4 kg spacecraft is launched from Earth and is to be radially accelerated away from the Sun using a circular solar sail. The initial acceleration of the sail is to be 1g. Assuming a flat sail, determine the radius of the sail if it is
(i) black, so it absorbs the sun's light (solar constant= $1.3\ kW/m^2$)
(ii) shiny, so it reflects the sun's light.

(b) A small part of the solar sail collects radiation emanating from an extremely large number of atoms in the solar photosphere, and these sources are incoherent. At any given instant, there is a near-infinite number of phasors in the complex plane, all of different magnitudes and different phases, rotating at different velocities. Their vector sum is zero, so perhaps the net field on the sail should be zero. How do you explain this paradox?

(c) Given the electromagnetic wave

$$\vec{E} = \hat{\mathbf{i}} E_o cos(kz - \omega t) + \hat{\mathbf{j}} E_o sin(kz - \omega t) \ ,$$

where E_o is a constant. Find the corresponding magnetic field $\vec{B}$ and the Poynting vector.

(d) Show that in free space with $\rho = 0, \ J = 0$, Maxwell's equations are correctly obtained from a single vector function $\vec{A}$ satisfying

$$\nabla \cdot \vec{A} = 0 \quad \text{and} \quad \nabla^2 \vec{A} - \tfrac{1}{c^2} \tfrac{d^2 \vec{A}}{dt^2} = 0 \quad .$$

(e) As shown by Arthur Eddington, there is a simple limit to the luminosity of a star or the X-ray emitting plasma around a neutron star or black hole. The limit derives from the case where the radiation pressure on an electron in the plasma is greater than the gravitational pull of the central astrophysical object. Assuming that the plasma is uniformly spherically distributed around the central object of mass M, derive an expression for this limiting luminosity, the Eddington limit, in terms of subatomic parameters. (Note: the luminosity os defined as the energy per unit time radiated off by an astrophysical object. Also, you may find it useful to know the classical electron radius,

$$r_e \equiv \left(\frac{e^2}{m_e c^2} \right) = 2.8 \times 10^{-13} \text{ cm}^2$$

and/or the Thomson scattering cross section, $\sigma_T = \frac{8\pi}{3} r_e^2 = 6.65 \times 10^{-25} \text{ cm}^2$.)

III. Mathematical Methods

Suppose we are interested in studying the motion of an an-harmonic oscillator whose Lagrangian is given by

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + g(t)x^3. \tag{1}$$

We will assume that $g(t)$ is small and acts over time δt which is much shorter than the time scale $1/\omega_0 = \sqrt{m/k}$. That is, we will treat the tri-linear term as a perturbation. We wish to find the motion of the oscillator to linear order in g at a time well after the perturbation is turned off. To do this we will use the Green's function method.

(a) Write down the equations of motion.

(b) The Green's function obeys

$$\ddot{G}(t-t') + \omega_0^2 G(t-t') = \delta(t-t') \ . \tag{2}$$

Show that the motion of x is given by

$$x(t) = x_0(t) + \frac{3}{m}\int dt' G(t-t')g(t')x^2(t') \ , \tag{3}$$

where $x_0(t)$ is a solution to the simple harmonic oscillator motion (i.e., ignoring the an-harmonic term).

(c) Write

$$G(t-t') = \frac{1}{2\pi}\int d\omega e^{i\omega(t-t')}\tilde{G}(\omega) \tag{4}$$

and solve for $\tilde{G}(\omega)$.

(d) Now we would like to solve for $G(t-t')$ but we find that the integral is not well defined due to the existence of poles on the real axis. To deal with this issue we will add an infinitesimal contribution to the action such that

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}(k+i\epsilon)x^2 + g(t)x^3, \tag{5}$$

where $\epsilon > 0$. How does this addition to the Lagrangian change eq. (2) ?

(e) This prescription makes the integral for $G(t-t')$ well defined. Perform the integration for the cases (a) $t > t'$, (b) $t < t'$.

(f) To leading order in g write down an equation for the motion of the oscillator assuming that the motion of the oscillator prior to the turning on of the perturbation is given by $x_0(t)$.