

# Graduate Qualifying Exam, Spring'08

## DAY TWO – Modern Physics

February 12, 2008

Please read carefully before starting –

On this second day of the exam we ask you to work three problems, each of which has several parts.

**Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.**

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

**Physical constants** (which you may find useful although you may not need all of them)

$$\epsilon_0 = 8.854 \times 10^{-12} \text{ C}^2/\text{N m}^2$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ N s}^2/\text{C}^2$$

$$c = 1/\sqrt{\epsilon_0\mu_0} = 3.0 \times 10^8 \text{ m/s}$$

$$e = 1.602 \times 10^{-19} \text{ C}$$

$$h = 6.626 \times 10^{-34} \text{ J s}$$

$$\hbar = h/2\pi = 6.582 \times 10^{-22} \text{ MeV s}$$

$$\hbar c = 197 \text{ eV nm} = 197 \text{ MeV fm}$$

$$(\hbar c)^2 = 0.389 \text{ GeV}^2 \text{ mbarn}$$

$$m_e = 0.511 \text{ MeV}/c^2$$

$$m_{\text{proton}} = 938 \text{ MeV}/c^2$$

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$$

$$1 \text{ eV}/c^2 = 1.783 \times 10^{-36} \text{ kg}$$

$$k_B = 1.381 \times 10^{-23} \text{ J/K}$$

$$N_A = 6.022 \times 10^{23} \text{ mol}^{-1}$$

$$1 \text{ Mpc} = 3.086 \times 10^{22} \text{ m}$$

$$G_N = 6.674 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$$

$$\mu_{\text{Bohr}} = 9.3 \times 10^{-23} \text{ J/T}$$

#### IV. Quantum Mechanics

To get full credit you must show ALL your work. None of the parts need more than a few lines of calculation. If you find yourself in a lengthy calculation then move on.

(a) Consider a particle moving in a one dimensional potential

$$V = \kappa x^n \quad (1)$$

Prove the Virial theorem which states that, in an energy eigenstate

$$2 \langle T \rangle = n \langle V \rangle \quad (2)$$

*HINT*: consider the commutator  $[xp, H]$ .

(b) Let  $e$  and  $m_e$  denote, respectively, the charge and the mass of an electron. Using the fact that the previous result generalizes to three-dimensional potentials use the uncertainty principle to show that the Bohr radius  $r_0$  is proportional to  $e^a m_e$  and that the Rydberg is proportional to  $e^b m_e$ , and determine the constants  $a, b$ . Suppose the nucleus generating the potential contains  $Z$  protons. Determine how the typical velocity of the electron scales with  $Z$ , that is if  $v \propto Z^c$  calculate  $c$ .

(c) Now we would like to understand the size of the energy level splittings. The moving electron feels a magnetic field due to motion in the Coulomb potential. The Hamiltonian for this interaction is given by

$$H_I = \frac{Ze^2}{2m^2c^2} \frac{1}{r^3} \vec{L} \cdot \vec{S} \quad (3)$$

Write down the complete set of commuting observables (you may neglect the proton) which also commute with the total Hamiltonian.

(d) Consider an eigenstate of the unperturbed Hamiltonian (i.e. neglecting  $H_I$ ) with  $L = 1$  and  $n = 2$ . Without including the spin orbit perturbation, write down the degeneracy of this state. Calculate the energy level splitting between these states due to  $H_I$ . Mark the degeneracy of each state. Leave your final answer for the energy in terms of  $\langle \frac{1}{r^3} \rangle$ .

(e) Repeat part (d) for the case where instead of a spin 1/2 electron bound to the potential, we have a spin one particle.

(f) Calculate  $\langle n | \frac{1}{r^3} | n \rangle$ . *HINT*: Use the fact that in the angular momentum basis

$$\frac{\partial H}{\partial r} = -\frac{\hbar^2 l(l+1)}{mr^3} + \frac{e^2}{r^2} \quad (4)$$

and

$$\langle n | \frac{1}{r^2} | n \rangle = \frac{2}{r_0^2 n^3 (2l+1)} \quad (5)$$

## V. Statistical Mechanics and Thermodynamics

Consider a one-dimensional gas of  $N + 1$  classical distinguishable particles. The particles cannot pass through each other, so we can number them from the left from 0 to  $N$ . Let the locations of the particles be given by the variables  $\{x_j | j = 0 \dots N\}$ , where  $x_{j+1} > x_j$ . Fix the position of the the first particle at  $x_0 = 0$ . The length of the system is clearly  $L = x_N$ .

The particles interact only with their nearest neighbors, but there is also a pressure term. The Hamiltonian is given by

$$H = \sum_{j=1}^N p_j^2/2m + A \sum_{j=1}^N (x_j - x_{j-1})^q + P x_N \quad (6)$$

where  $A$ ,  $q$ , and  $P$  are constants.  $A$  is the strength of the interactions and  $P$  is the pressure. The momenta are denoted by  $\{p_j | j = 0 \dots N\}$ , and the mass of each particle is  $m$ .

Let the temperature be denoted as  $T$  and let  $\beta = 1/k_B T$ .

- (a) Write an expression for the canonical partition function,  $Z$ .
- (b) Simplify your answer to part (a) by evaluating the momentum integrals.
- (c) Find an expression for the length of the system in terms of a derivative of the partition function with respect to an appropriate variable.
- (d) For  $q = 1$ , evaluate  $Z$ .
- (e) Again for  $q = 1$ , find the length of the system.
- (f) For  $P = 0$  but arbitrary  $q$ , find the specific heat per particle.

## VI. General Physics

The 'gegenschein' is a very faint, extended glow that can sometimes be seen in the summer sky around midnight at a position in the sky directly opposite the sun's position below the horizon; a similar glow, the zodiacal light, can also be seen about an hour after sunset, as a band extending along the ecliptic (annual path of the sun through the sky). These glows are caused by light reflection by dust in the solar system - in the case of the gegenschein, the light is back-scattered. Consider a dust particle in orbit around the sun, and ignore the effects of the solar wind and magnetic field.

(a) Estimate the minimum size that a dust particle must have to avoid being blown out of the solar system. Assume that a dust particle has a density of about  $10^3 \text{ kg m}^{-3}$

(b) Particles larger than the minimum size will spiral into the sun. What is the source of this drag? Estimate the lifetime of such a particle in orbit at the distance of the earth. [*HINT*: ignore the effects in part (a) and consider the effects of absorption and re-emission of radiation.]

$$M_{Sun} = 2 \times 10^{30} \text{ kg}$$

$$R_{earth} = 1.5 \times 10^8 \text{ km}$$

$$G = 6.67 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2}$$

$$L_{sun} = L_{\odot} = 4 \times 10^{26} \text{ W}$$