Graduate Qualifying Exam, Spring'08

DAY ONE – Classical Physics

February 11, 2008

Please read carefully before starting –

On this first day of this exam you are asked to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

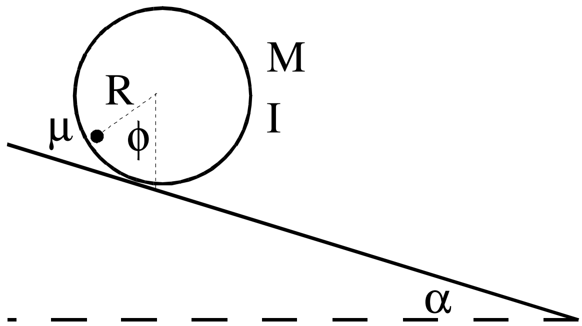
Note that you are expected to work all the problems covered in the exam.

Physical constants (which you may find useful although you may not need all of them)

$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N\,m^2}$	$\mu_0 = 4\,\pi \times 10^{-7}\ \mathrm{N\,s^2/C^2}$
$c = 1/\sqrt{\epsilon_0\mu_0} = 3.0 \times 10^8\ \mathrm{m/s}$	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
$h = 6.626 \times 10^{-34}\ \mathrm{J\,s}$	$\hbar = h/2\pi = 6.582 \times 10^{-22}\ \mathrm{MeV\,s}$
$\hbar c = 197\ \mathrm{MeV\ fm} = 197\ \mathrm{eV\ nm}$	$(\hbar c)^2 = 0.389\ \mathrm{GeV^2\ mbarn}$
$m_e = 0.511\ \mathrm{MeV}/c^2$	$m_{proton} = 938\ \mathrm{MeV}/c^2$
$m_\mu = 105.7\ \mathrm{MeV}/c^2$	$m_{\pi^0} = 135\ \mathrm{MeV}/c^2$
$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$	$1\ \mathrm{eV}/c^2 = 1.783 \times 10^{-36}\ \mathrm{kg}$
$k_B = 1.381 \times 10^{-23}\ \mathrm{J/K}$	$N_A = 6.022 \times 10^{23}\ \mathrm{mol^{-1}}$
$G_N = 6.674 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$	

I. Classical Mechanics

1) A wheel has mass M, radius R, and moment of inertia I (about its center). An additional point mass, μ, is then added at the edge of the wheel. The wheel rolls without slipping on a plane which is inclined at angle α. Use as a coordinate the angle, ϕ, which is the angle of the point mass measured (clockwise) from a **vertical** line dropped from the center of the wheel. (The gravitational field, g, is constant and the wheel rolls only in the plane of the diagram.)



(a) Find the total potential energy as a function of ϕ. (Be careful to correctly determine the positions of the wheel and of the point mass.) Sketch the potential vs. ϕ.

(b) Find <u>all</u> positions of equilibrium and classify them as stable or unstable. If the existence or stability of some equilibrium positions depends on the values of the parameters, give the required conditions.

(c) If stable equilibrium positions do exist, what initial condition(s) must be satisfied for the motion of the system to be <u>bounded</u>? Your sketch of the potential may be helpful. (There may be many stable equilibrium positions, try to describe the required initial conditions, in general, for bounded motion near any of them.)

(d) Carefully determine the total kinetic energy of the system. (Does the kinetic energy of the point mass have the expected behavior for $\phi = \alpha$?) Write down the Lagrangian of the system.

(e) Write down Lagrange's equation(s) of motion and determine the frequency of **<u>small</u>** oscillations about an equilibrium position. (You may use the equilibrium position, ϕ_0, in your answer.)

(f) Find p_ϕ,the momentum conjugate to ϕ. **You don't need to find the Hamiltonian to answer these questions.** Consider the Hamiltonian written in terms of ϕ and p_ϕ. i)Is the Hamiltonian of this system the total energy? ii)Is the Hamiltonian conserved? What would be the answers to i) and ii) be if the angle, α, of the ramp were being slowly and steadily decreased during the motion of the system?

II. Electromagnetism

An isolated conducting sphere of radius a is located with its center at a distance z from a grounded infinite conductor plate. Assume that $z >> a$ and find:

(a) the leading contribution to the capacitance in a/z between the sphere and the plane;

(b) the first (non-vanishing) correction to this value, when the capacitance is expressed in terms of a power-series expansion in a/z;

(c) to leading order (in a/z) the force between the sphere and plane when the sphere carries a charge Q.

(d) If the sphere carries charge Q, what is the energy required for complete separation of the sphere from the plane? How does this energy compare with the energy required for separating two spheres with charges $+Q$ and $-Q$, initially spaced apart by a distance $2z$? Explain any differences (if any) between these two energy values.

III. Mathematical Physics

The circular head of a drum obeys the wave equation

$$\nabla^2 h = \frac{1}{v^2}\frac{\partial^2 h}{\partial t^2} \tag{1}$$

where $h(r, \phi, t)$ is the space- and time-varying displacement, and v is the speed of wave propagation. Note that the Laplacian operator in cylindrical polar coordinates is

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} \tag{2}$$

(a) In a normal mode the entire drum head oscillates at a single frequency ω. Find the functional form of a typical normal mode. Your solution should include the Bessel function $J_m(kr)$ which is analytic at the origin (for m a positive integer) and solves the differential equation

$$x^2\frac{d^2}{dx^2}J_m(x) + x\frac{d}{dx}J_m(x) + (x^2 - m^2)J_m(x) = 0. \tag{3}$$

(b) Boundary conditions require that h vanish at the edge of the drum, $r = a$. Use this fact to determine all allowed normal mode frequencies. Be sure to fully define any quantities or symbols you introduce in your answers. You may express your answers in terms of the zeros of the Bessel function.

(c) Calculate the limiting form of $J_m(x)$ for $m > 0$ in the limit of small x.

(d) The function $J_0(x) \to 1$ as $x \to 0$. Calculate the lowest order correction for small x.

(e) Determine the asymptotic form of $J_0(x)$ for large x.