Graduate Qualifying Exam, Spring'07

DAY TWO – Modern Physics

February 16, 2007

Please read carefully before starting –

On this second day of the exam we ask you to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

Physical constants (which you may find useful although you may not need all of them)

$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N\,m^2}$ $\qquad$ $\mu_0 = 4\,\pi \times 10^{-7}\ \mathrm{N\,s^2/C^2}$

$c = 1/\sqrt{\epsilon_0 \mu_0} = 3.0 \times 10^8\ \mathrm{m/s}$ $\qquad$ $e = 1.602 \times 10^{-19}\ \mathrm{C}$

$h = 6.626 \times 10^{-34}\ \mathrm{J\,s}$ $\qquad$ $\hbar = h/2\pi = 6.582 \times 10^{-22}\ \mathrm{MeV\,s}$

$\hbar c = 197\ \mathrm{eV\ nm} = 197\ \mathrm{MeV\ fm}$ $\qquad$ $(\hbar c)^2 = 0.389\ \mathrm{GeV^2\,mbarn}$

$m_e = 0.511\ \mathrm{MeV}/c^2$ $\qquad$ $m_{proton} = 938\ \mathrm{MeV}/c^2$

$m_\mu = 105.7\ \mathrm{MeV}/c^2$ $\qquad$ $m_{\pi^0} = 135\ \mathrm{MeV}/c^2$

$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$ $\qquad$ $1\ \mathrm{eV}/c^2 = 1.783 \times 10^{-36}\ \mathrm{kg}$

$k_B = 1.381 \times 10^{-23}\ \mathrm{J/K}$ $\qquad$ $N_A = 6.022 \times 10^{23}\ \mathrm{mol^{-1}}$

$1\ \mathrm{Mpc} = 3.086 \times 10^{22}\ \mathrm{m}$ $\qquad$ $G_N = 6.674 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$

IV. Quantum Mechanics

To get full credit you must show ALL your work. None of the parts in this problem need more than a few lines of calculation. If you find yourself in a lengthy calculation then stop and think.

A rigid rotator has the Hamiltonian $H = \frac{\vec{L}^2}{2I} + \kappa L_z$, where I and κ are constants.

(a) Write down a complete set of commuting observables for this system.

(b) What are the energy eigenvalues if the total angular momentum is $l = 3$?

(c) At $t = 0$ the system is in the state $\mid \psi\rangle = \frac{1}{\sqrt{2}}(\mid L = 1, m = 1\rangle + \mid L = 1, m = -1\rangle)$. Calculate $\langle L_y\rangle(t)$.

(d) If the system is in a state with $l = 1, m = 1$, calculate the uncertainty ΔL_x.

(e) Prove the following identity,

$$-i\hbar \frac{d}{dt}\langle O\rangle = \langle [H, O]\rangle \ ,$$

where O is some Hermitian operator satisfying $\frac{\partial O}{\partial t} = 0$.

(f) Calculate $\frac{d}{dt}\langle L_i\rangle$ in terms of expectation values of other operators for $i = x$ and $i = y$.

(g) Use the results of the previous problem to determine $\langle L_x\rangle$ for all times assuming that $\frac{d}{dt}\langle L_x\rangle = 0, \langle L_x\rangle = l$, at $t = 0$. What kind of motion does this correspond to?

(h) Now consider two independent identical rigid rotators, with identical moments of inertia. The angular momentum of rotator one is in the state $L = 1, m = 1$ while the second is in the state $L = 1, m = 1$. What is the probability of finding the total angular momentum to be 1 ?

V. Statistical Mechanics and Thermodynamics

A magnetic system consists of $\mathcal{N}$ spins located on lattice sites i, whose states can be described by two quantum numbers representing the total angular momentum, $J = 1$, and its z component, m_i takes on the values $-1, 0$, or 1. In a magnetic field of strength B directed along the z axis, the spin states have energies

$$H = \mu_0 B \ \Sigma_i m_i,$$

where $\mu_0 > 0$ is a magnetic moment.

(a) Use the canonical ensemble to compute the Helmholtz free energy $F = U - TS$ of this system as a function of temperature T and magnetic field strength.

(b) Compute the internal energy U and the magnetization

$$M = -\left(\frac{\partial F}{\partial B}\right)_T$$

of the system and make a well-labeled sketch of each as a function of temperature.

(c) Compute the entropy of the system and give analytical forms for its leading non-vanishing values at very low and very high temperatures.

(d) Suppose that the energy of the system were to be modified in the following way to include the effects of a crystal field:

$$H = \Sigma_i(\mu_0 B m_i + D m_i^2)$$

Assume that you had already derived the free energy, F for this model. Derive a thermodynamic derivative of the free energy that gives the thermal average

$$< \Sigma_i m_i^2 > .$$

(e) Some elementary books use the formula

$$M = -U/B,$$

to compute the magnetization from the internal energy. As can be seen from the results in part (b), this equation is sometimes valid. Demonstrate whether this equation remains valid for the spin state energy given in part (d).

VI. General Physics

Consider a free proton in an uniform external magnetic field given by $\vec{B} = B_0 \hat{\mathbf{z}}$, where $B_0 = 2.0\ Tesla$.

(a) Suppose the proton has initial momentum $\vec{p} = 100.0\ \hat{\mathbf{x}}$ MeV/c. Estimate the orbital radius (in *meters*) and frequency (in Hz) of the proton in the field. Show whether Special Relativity is relevant here or not.

(b) Compare the magnitudes of the cyclotron frequency, ω_c, of a classical charge e and mass m with the angular momentum (Larmor) precession frequency, ω_p, of a classical magnetic moment characterized by the same e and m.

(c) Next suppose the proton in Part (a) is initially spin polarized along $\hat{\mathbf{x}}$. Estimate the orientation of the proton's spin after one complete orbital revolution in the magnetic field. The proton is not a classical object: it has a structure factor of $g = 5.585$.

(d) Suppose there is an Avogadro's number of protons at rest in the field (*i.e.* a macroscopic amount of material), all initially aligned along $\hat{\mathbf{x}}$, and you wish to detect the spin precession of the protons via magnetic induction in a pick-up coil. Carefully draw and explain a diagram showing how you could orient this coil with respect to the given coordinates and field in order to detect a signal.

(e) Now suppose the proton is *at rest* in the $\vec{B}$ field, but with its spin oriented (polarized) along $\hat{\mathbf{z}}$, parallel to $\vec{B}$. Let there be an additional component to the magnetic field given by $\vec{B_1} = B_1 \cos(\omega t)\hat{\mathbf{x}}$ where $B_1 = 1. \times 10^{-4}$ Tesla and ω is a variable angular frequency. Discuss the subsequent motion of the proton and its polarization vector in the limit that ω is the Larmor frequency for the proton in the $\vec{B}$ field. In particular, when is the probability maximized that the proton can be measured with its spin down (along $-\hat{\mathbf{z}}$)?