Graduate Qualifying Exam, Spring'06

DAY TWO – Modern Physics

February 10, 2006

Please read carefully before starting –

On this second day of the exam we ask you to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

Physical constants (which you may find useful although you may not need all of them)

$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N\,m^2}$	$\mu_0 = 4\,\pi \times 10^{-7}\ \mathrm{N\,s^2/C^2}$
$c = 1/\sqrt{\epsilon_0 \mu_0} = 3.0 \times 10^8\ \mathrm{m/s}$	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
$h = 6.626 \times 10^{-34}\ \mathrm{J\,s}$	$\hbar = h/2\pi = 6.582 \times 10^{-22}\ \mathrm{MeV\,s}$
$\hbar c = 200\ \mathrm{MeV\,fm}$	$(\hbar c)^2 = 0.389\ \mathrm{GeV^2\,mbarn}$
$m_e = 0.511\ \mathrm{MeV}/c^2$	$m_p = 938.3\ \mathrm{MeV}/c^2$
$m_\mu = 105.7\ \mathrm{MeV}/c^2$	$m_{\pi^0} = 135\ \mathrm{MeV}/c^2$
$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$	$1\ \mathrm{eV}/c^2 = 1.783 \times 10^{-36}\ \mathrm{kg}$
$k_B = 1.381 \times 10^{-23}\ \mathrm{J/K}$	$N_A = 6.022 \times 10^{23}\ \mathrm{mol^{-1}}$
$1\ \mathrm{Mpc} = 3.086 \times 10^{22}\ \mathrm{m}$	$G_N = 6.674 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$

IV. Quantum Mechanics

A particle of mass m is in a 1-d harmonic oscillator potential

$$V = \frac{1}{2} m\omega^2 x^2$$

(a) Show that the Hamiltonian can be written as

$$H = \hbar\omega(a^\dagger a + 1/2)$$

where $a, a^\dagger$ are the raising and lower operators respectively. Write these operators in terms of x and p.

(b) Show that

$$\begin{aligned} a^\dagger \mid n\rangle &= \alpha_+(n) \mid n+1\rangle \\ a \mid n\rangle &= \alpha_-(n) \mid n-1\rangle, \end{aligned}$$

and evaluate α_+, α_-. You can assume that the eigenvalues of the Hamiltonian are $\hbar\omega(n+1/2)$ with eigenvectors $\mid n\rangle$, where n is an integer greater than zero.

(c) Now consider adding to the system another (distinguishable) particle of equal mass whose potential is given by

$$V(x_2) = \frac{1}{2} m\omega^2 x_2^2.$$

The particles are coupled via a perturbing force

$$V_{12} = \frac{1}{4} m\Omega^2 (x_1 - x_2)^2.$$

Use perturbation theory to calculate the first order correction to the ground state energy of the system.

(d) Use first order perturbation theory to calculate the energies of the first two excited states. What are the corresponding eigenvectors?

(e) At $t = 0$ the system is prepared such that particle one is in the state $\mid 1\rangle$ and particle two is in the state $\mid 0\rangle$. What is the probability of finding particle one in the (unperturbed) state $\mid 1\rangle$ as a function of time?

V. Statistical Mechanics

Consider an ideal classical gas of N atoms confined to a container of volume V and internal surface area A. Although you may neglect the interactions between atoms, there is an attraction between the atoms and the walls of the container that cannot be ignored. A simple model for the atoms adsorbed onto the surface is to treat them as a two-dimensional classical ideal gas, where the energy of an adsorbed atom is

$$\varepsilon(\vec{p}) = \frac{|\vec{p}|^2}{2m} - \varepsilon_o$$

and $\vec{p}$ is the two-dimensional momentum. Do not concern yourself with the details of the binding; treat ε_o as a known parameter.

The entire system is in thermal equilibrium with a heat reservoir at temperature T.

(a) What is the classical partition function of the adsorbed atoms if N' of them are bound to the surface?

(b) What is the chemical potential μ_s of the adsorbed atoms?

(c) What is the classical partition function of the $N - N'$ atoms in the volume of the container?

(d) What is the chemical potential of the $N - N'$ atoms in the volume of the container?

(e) When the atoms in the volume and those on the surface are in equilibrium with each other, what is the average number of atoms adsorbed as a function of the temperature T ?

(f) How many atoms are adsorbed on the walls in the limits of high and low temperatures, according to your answer to (e)? Does your answer make sense in these two limits?

VI. General Physics

(a) For a gas consisting entirely of *neutral* hydrogen atoms, at what temperature T is the number of atoms in the first excited state (i) 1% and (ii) 10% of the number of atoms in the ground state?

Consider a box of *electrically neutral* hydrogen gas that is maintained at a constant volume V. The number of electrons must equal the number of ionized ions (H II ions): $n_e V = N_{II}$. Also, the total number of hydrogen atoms (both neutral and ionized), N_t, is related to the density of the gas by

$$N_t = \rho V/(M_p + m_e) \simeq \rho V/m_p$$

where m_p is the mass of the proton (the mass of the electron is ignored). Let the density of the gas be 10^{-9} g cm^{-3}, typical of the photosphere of an average star.

(b) Make these substitutions into the Saha equation (see below) to derive the quadratic equation for the fraction of ionized atoms,

$$(N_{II}/N_t)^2 + (N_{II}/N_t)(m_p/\rho)(2\pi m_e kT/h^2)^{3/2} e^{-\chi_I/kT} - (m_p/\rho)(2\pi m_e kT/h^2)^{3/2} e^{-\chi_I/kT} = 0 \ .$$

(c) Solve the above quadratic equation for the fraction of ionized hydrogen, N_{II}/N_t for a range of temperatures between 5000 K and 25,000 K – sketch a very rough graph of the results.

(d) Use the Saha equation to determine the fraction of hydrogen atoms that are ionized, N_{II}/N_{total} at the center of the sun, where the temperature is about 17 million K and the number density of electrons is about $n_e = 6.2 \times 10^{25}$ cm^{-3} (use $Z_I = 2$). Does your result agree with the fact that practically all of the sun's hydrogen is ionized at the sun's center? What is the reason for any discrepancy?

Note: Saha equation:

$$\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{n_e Z_i} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \exp\left(\frac{-\chi_i}{kT}\right)$$

where Z_i is the number of possible degenerate states (*e.g.*, 2 for the ground-state hydrogen atom), and χ_i is the ionization potential.