Graduate Qualifying Exam, Spring'06

DAY ONE – Classical Physics

February 9, 2006

Please read carefully before starting –

On this first day of this exam you are asked to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

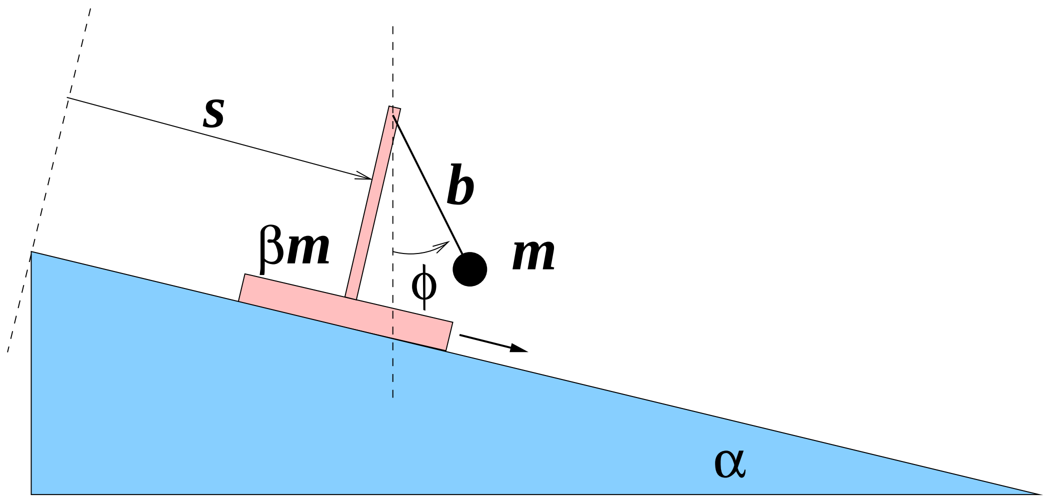
In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

Physical constants (which you may find useful although you may not need all of them)

$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N\,m^2}$	$\mu_0 = 4\,\pi \times 10^{-7}\ \mathrm{N\,s^2/C^2}$
$c = 1/\sqrt{\epsilon_0\mu_0} = 3.0 \times 10^8\ \mathrm{m/s}$	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
$h = 6.626 \times 10^{-34}\ \mathrm{J\,s}$	$\hbar = h/2\pi = 6.582 \times 10^{-22}\ \mathrm{MeV\,s}$
$\hbar c = 200\ \mathrm{MeV\,fm}$	$(\hbar c)^2 = 0.389\ \mathrm{GeV^2\,mbarn}$
$m_e = 0.511\ \mathrm{MeV}/c^2$	$m_p = 938.3\ \mathrm{MeV}/c^2$
$m_\mu = 105.7\ \mathrm{MeV}/c^2$	$m_{\pi^0} = 135\ \mathrm{MeV}/c^2$
$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$	$1\ \mathrm{eV}/c^2 = 1.783 \times 10^{-36}\ \mathrm{kg}$
$k_B = 1.381 \times 10^{-23}\ \mathrm{J/K}$	$N_A = 6.022 \times 10^{23}\ \mathrm{mol^{-1}}$
$1\ \mathrm{Mpc} = 3.086 \times 10^{22}\ \mathrm{m}$	$G_N = 6.674 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$

I. Classical Mechanics



A pendulum of mass m and length b is mounted to a heavy sled of mass βm which is free to slide with negligible friction down an inclined plane of slope α, as shown. The entire system acts under the influence of gravity, where g is the downward acceleration due to gravity.

(a) Let ϕ denote the angle that the pendulum bob makes with the vertical, and s denote the distance the sled has slid down the slope (see diagram). Determine the Lagrangian $L(\phi, s, \dot{\phi}, \dot{s})$ describing the system in terms of ϕ and s and their rates of change $\dot{\phi}$ and $\dot{s}$ with respect to time.

(b) Find expressions for the momenta p_ϕ and p_s conjugate to ϕ and s, respectively.

(c) Describe the steps you would follow to find the Hamiltonian $H(\phi, s, p_\phi, p_s)$ of the system. Do *not* actually calculate H, just describe how you would determine it.

(d) From the equations of motion for the system, find expressions for the generalized accelerations $\ddot{\phi}$ and $\ddot{s}$ in terms $\phi, s, \dot{\phi}, \dot{s}$.

(e) The sled can slide down the slope with the pendulum at an unchanging angle ϕ_0 if ϕ_0 satisfies a certain condition. Determine this condition.

II. Electromagnetism

Consider an azimuthally symmetric magnetic field, such as might be produced by a dipole magnet with circular pole faces, of the form $\vec{B} = B(r)\hat{z}$. An electron of charge e orbits in this field at a fixed radius R and with momentum $\vec{p}$ in the (r, θ) plane.

(a) What is the orbital frequency of the electron, ω_0, assuming the magnetic field is time independent?

(b) Suppose we inductively accelerate the electron by varying the magnetic field slowly in time in some way. We want the electron's radius of orbit to remain fixed at R. Derive a condition that relates $B(R)$ and the average value of the field within the orbit, $< B >$, that allows this to happen. (Note that this is the so-called Betatron condition.) Make a rough sketch of $B(r)$ vs. r that allows this acceleration technique to work.

(c) What is the change in momentum of the electron if the magnetic field scale starts at value B_0 and ends at value B_1 ?

In order for the electron orbit to be stable, it must be stable against small displacements in the radial (r) and axial (z) directions. Let the magnetic field near the ideal orbit at radius R have components B_z and B_r, though we require $B_r(z = 0) = 0$.

(d) Write the equations of motion for the electron in terms of these two components in cylindrical variables.

(e) In the vicinity of the orbiting electron at radius R, let the z component of the field be given by $B_z(r) = B_0(r/R)^{-n}$, where n is the field index with $0 < n < 1$. What is the corresponding radial component of the field as a function of z ?

(f) Using suitable linearization of the equations for small displacements, show that the oscillation frequencies of the electron are $\omega_r = (1 - n)^{1/2}\omega_0$ and $\omega_z = n^{1/2}\omega_0$ for the radial and the axial oscillations, respectively.

III. Mathematical Physics

This problem has to do with the diffusion of heat in a slab of material. For the purposes of the problem we will ignore thermal expansion. As such we may define the internal energy density in terms of the heat capacity per unit mass c_p such that the total energy of the sample is given by

$$E = \int d^3r c_p \rho(\vec{r}, t) T(\vec{r}, t)$$

where ρ is the mass density and T is temperature. The "heat current density" is defined via

$$\vec{j} = -k_{th} \vec{\nabla} T(\vec{r}, t).$$

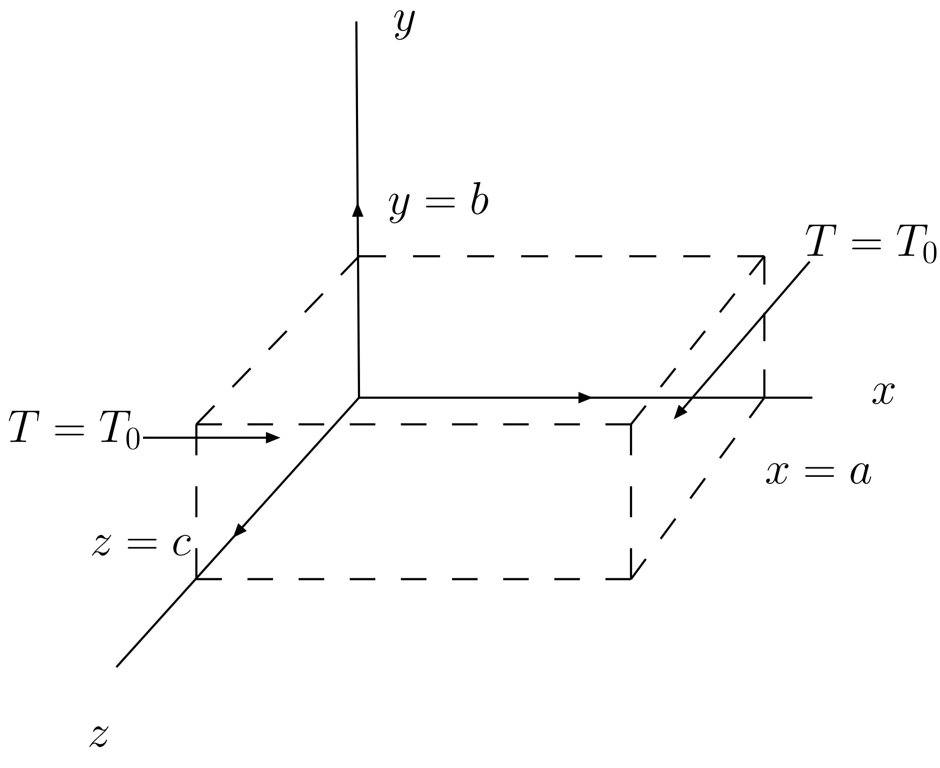
where k_{th} is the "thermal conductivity".

(a) Assuming that there are no sources or sinks and that the density is constant, derive the heat diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T,$$

expressing the diffusivity constant (κ) in terms of c_p , k_{th}, and ρ. What are the units of κ ?

(b) Consider a slab of material with diffusivity κ that has a uniform mass distribution $\rho = \rho_0$. The geometry is such that at $x = 0$ and at $x = a$ the slab is held at fixed temperature T_0. The remaining sides are insulated (no heat flow in or out). The situation is depicted in the picture shown below.



At $t = 0$ the temperature is given by $T(\vec{r}, 0)$. What are the appropriate spatial boundary conditions?

(the Mathematical Physics problem, continued from previous page)

(c) To solve for the temperature as a function of time it is best to write the diffusion equation in terms of the variable

$$\delta T = T - T_0$$

such that when $t \to \infty$ we have $\delta T \to 0$. Using the ansatz

$$\delta T = \delta T(\vec{r}) e^{-\lambda t}$$

derive an equation of the form

$$(\nabla^2 + \lambda/\kappa)\delta T(\vec{r}) = 0$$

and write down an expression (in terms of an infinite sum) for all possible values of λ .

(d) Show that in the long time limit only one term of all the possible values in the above sum is relevant and determine that value.

(e) Write down the time dependent expression for the temperature in terms of the initial distribution $T(\vec{r}, 0)$ in the long time limit (keeping the first deviation from infinite time limit $T = T_0$). The final expression should have an integral remaining to be done over $T(\vec{r}, 0)$.