Graduate Qualifying Exam, Fall'07

DAY TWO – Modern Physics

August 22, 2007

Please read carefully before starting –

On this second day of the exam we ask you to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

Physical constants (which you may find useful although you may not need all of them)

$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N\,m^2}$	$\mu_0 = 4\,\pi \times 10^{-7}\ \mathrm{N\,s^2/C^2}$
$c = 1/\sqrt{\epsilon_0\mu_0} = 3.0 \times 10^8\ \mathrm{m/s}$	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
$h = 6.626 \times 10^{-34}\ \mathrm{J\,s}$	$\hbar = h/2\pi = 6.582 \times 10^{-22}\ \mathrm{MeV\,s}$
$\hbar c = 197\ \mathrm{eV\ nm} = 197\ \mathrm{MeV\ fm}$	$(\hbar c)^2 = 0.389\ \mathrm{GeV^2\,mbarn}$
$m_e = 0.511\ \mathrm{MeV}/c^2$	$m_{proton} = 938\ \mathrm{MeV}/c^2$
$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$	$1\ \mathrm{eV}/c^2 = 1.783 \times 10^{-36}\ \mathrm{kg}$
$k_B = 1.381 \times 10^{-23}\ \mathrm{J/K}$	$N_A = 6.022 \times 10^{23}\ \mathrm{mol^{-1}}$
$1\ \mathrm{Mpc} = 3.086 \times 10^{22}\ \mathrm{m}$	$G_N = 6.674 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$
$\mu_{Bohr} = 9.3 \times 10^{-23}\ \mathrm{J/T}$	

IV. Quantum Mechanics

To get full credit you must show ALL of your work. None of these problems need more than a few lines of calculation. If you find yourself in a lengthy calculation move on.

(a) Consider an operator O . Given the following properties, write down what conditions O would have to satisfy– (i) O is Hermitian, (ii) O is unitary.

(b) Suppose that O is Hermitian. Prove that its eigenvalues are real.

(c) Suppose that $[O, H] = 0$ where H is the Hamiltonian for the system of interest. Show that

$$\langle \psi(t) \mid O \mid \psi(t) \rangle = C \tag{1}$$

where C is a constant. That is, show that the expectation value of O in any state is independent of time.

(d) Suppose that we have two Hermitian operators O_1 and O_2 that both commute with the Hamiltonian. Furthermore, suppose that $[O_1, O_2] \neq 0$ (this holds for all matrix elements). Show that given an energy eigenstate $\mid a\rangle$ such that

$$O_1 \mid a\rangle \neq 0 \tag{2}$$

this implies that $\mid a\rangle$ must be degenerate.

(e) Consider the Hydrogen atom with Hamiltonian

$$H = \frac{\vec{p}^2}{2m} - \frac{e^2}{r}. \tag{3}$$

Draw an energy level diagram, for the $n = 1$ and $n = 2$ levels, being sure to include spin. Mark all degeneracies as well as the energy of the states.

(f) For each degeneracy write down two operators which can be used to prove the existence of the degeneracy.

V. Statistical Mechanics and Thermodynamics

Consider a one-dimensional array of N spins, $\{\sigma_j | j = 1, \ldots, N\}$, where each spin can take on the values -1 or $+1$ (Ising spins).

The spins interact with each other and with a magnetic field. The Hamiltonian is given by

$$H = -J \sum_{j=1}^{N-1} \sigma_j \sigma_{j+1} - h \sum_{j=1}^{N} \sigma_j$$

where J is the interaction constant and h is the magnetic field. The magnetization of the system is $M = \sum_{j=1}^{N} \sigma_j$. The system is in equilibrium with a heat reservoir at temperature T (or inverse temperature $\beta = 1/kT$).

(a) Derive a general relation between the magnetic susceptibility,

$$\chi = \left(\frac{\partial \langle M \rangle}{\partial h} \right)_T ,$$

and the fluctuations of the magnetization, $\left\langle (M - \langle M \rangle)^2 \right\rangle$, that is valid for any values of J and h.

(b) For the case of $J = 0$, but $h \neq 0$, calculate the partition function and the average magnetization.

(c) For the case of $h = 0$ but arbitrary J, calculate the partition function, the average energy, and the entropy. What is the zero-temperature limit of the entropy when $J > 0$, and when $J = 0$?

VI. General Physics

Positronium is a hydrogen-like bound state made up of an electron and a positron.

(a) Estimate the binding energy of the ground state (n = 1) and the Lyman-alpha (2p $\rightarrow$ 1s) transition wavelength for positronium.

[*Hint:* $\alpha = 1/137$ *is the fine-structure constant, and a Rydberg is an energy unit with 1 Ry = 13.6 eV, the ground state energy of hydrogen.*]

(b) The lifetime for decay from 2p to 1s for the hydrogen atom is 1.6 ns. Use dimensional analysis or any other method to estimate the lifetime for the same decay in positronium.

[*Hint: The decay rate* Γ *for electric dipole transitions between two states is proportional to* $\omega^3 \mid < \mathbf{r} > \mid^2$, *where* $\hbar\omega$ *is the energy difference between the states,* $\mathbf{r}$ *is the relative coordinate between the positron and the electron, and the constant of proportionality depends only on natural constants and pure numbers.*]

(c) Estimate the strength of the magnetic field experienced by the electron due to the positron's magnetic moment for the $n = 1$ state.

(d) Estimate the singlet-triplet frequency splitting in the ground state.

[*Hint: Recall that the hyperfine splitting in the ground state of H gives rise to the 21 cm line, important in astronomy.*]

(e) The singlet state decays into two 0.511 MeV γ-rays in a state of defined parity. If the γ-rays are detected in coincidence after passing through linear polarizers (see figure), at what relative angle ϕ is a maximum coincidence rate expected?

[*Hint: The question of correlation of photon polarizations forms the basis for the Einstein-Podolsky-Rosen paradox. The parity of the singlet* ^1S *state is* $-1(-1)^{L+S} = -1$, *where the first factor of* -1 *is due to the opposite intrinsic parities of the electron and positron.*]