Graduate Qualifying Exam, Fall'07

DAY ONE – Classical Physics

August 21, 2007

Please read carefully before starting –

On this first day of this exam you are asked to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

Physical constants (which you may find useful although you may not need all of them)

$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N\,m^2}$

$\mu_0 = 4\,\pi \times 10^{-7}\ \mathrm{N\,s^2/C^2}$

$c = 1/\sqrt{\epsilon_0 \mu_0} = 3.0 \times 10^{8}\ \mathrm{m/s}$

$e = 1.602 \times 10^{-19}\ \mathrm{C}$

$h = 6.626 \times 10^{-34}\ \mathrm{J\,s}$

$\hbar = h/2\pi = 6.582 \times 10^{-22}\ \mathrm{MeV\,s}$

$\hbar c = 197\ \mathrm{MeV\ fm} = 197\ \mathrm{eV\ nm}$

$(\hbar c)^2 = 0.389\ \mathrm{GeV^2\ mbarn}$

$m_e = 0.511\ \mathrm{MeV}/c^2$

$m_{proton} = 938\ \mathrm{MeV}/c^2$

$m_\mu = 105.7\ \mathrm{MeV}/c^2$

$m_{\pi^0} = 135\ \mathrm{MeV}/c^2$

$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$

$1\ \mathrm{eV}/c^2 = 1.783 \times 10^{-36}\ \mathrm{kg}$

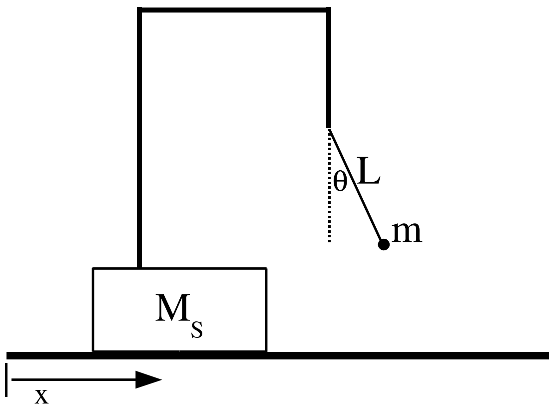
$k_B = 1.381 \times 10^{-23}\ \mathrm{J/K}$

$N_A = 6.022 \times 10^{23}\ \mathrm{mol^{-1}}$

$G_N = 6.674 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$

I. Classical Mechanics

1) A pendulum which is free to swing in the x-y plane is composed of a rigid rod of length L rigidly connected to a ball of mass m. The rod is of negligible mass while the ball is small enough to be treated as a point particle. The pendulum is supported from a perfectly rigid stand which has a total mass M_S (including the base and supports, but not including the pendulum). This support can slide **without friction** in the x direction. The angle between the pendulum and the vertical is θ, as shown, and the position of the stand is x.



a) Write down the Lagrangian of the system.

(b) Indicate any cyclic variables and identify the conserved quantities associated with them.

(c) Using the Lagrangian, find the equations of motion. Don't attempt to solve them. Find the explicit form of the conserved quantity (or quantities) identified in part b).

(d) Using the equations of motion, find the frequency of oscillation of the pendulum for small displacement. Show that your result simplifies to the expected result in the limit $M_S >> m$.

(e) Find the horizontal position of the center of mass, X_{CM}, of the system in terms of x and θ. (You may take x to be the position of the center of mass of the stand and take the position of the pivot to be $x + D$, where D is a constant offset.) What value do you expect for the second derivative of this position with respect to time, $\ddot{X}_{CM}$? Explain. Do your equations of motion predict the correct value for $\ddot{X}_{CM}$?

(f) If the small ball were replaced by a **large** solid spherical ball of mass m and radius R, how would the Lagrangian differ from that which you found in part a)? You may assume that the center of the ball is still a distance L away from the pivot and that the large ball is rigidly connected to the rod, but the ball is no longer small enough to be treated as a point particle. If you need a moment of inertia in the new Lagrangian, calculate it explicitly from first principles, don't use a memorized result. (Hint: you may find it useful to use the parallel-axis theorem for moments of inertia)

II. Electromagnetism

(1) Consider a uniformly charged conducting sphere with charge Q and radius a, surrounded by vacuum. Derive the boundary conditions that must be satisfied by the electrostatic field at the surface of the conducting sphere.

(2) Consider a point charge q located outside a grounded conducting sphere of radius a. Let $\vec{\mathbf{y}}$ denote the location of the point charge relative to the center of the sphere, with $|\vec{\mathbf{y}}| > a$. Use *Dirichlet* boundary conditions and the method of images to solve the following parts:

(a) Derive the appropriate Green's function for this configuration.

(b) What happens to the location and the magnitude of the image charge as q is brought closer to the surface of the sphere and as q moves toward infinity?

(c) Express the potential outside the conducting sphere as a sum of Legendre polynomials.

Hint: $\frac{1}{\vec{r}-\vec{r}\,'} = \sum\limits_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(cos\gamma)$, where $r_<$ and $r_>$ represent the smaller and larger of $|\vec{r}|$ and $|\vec{r}\,'|$, respectively and γ is the angle between $\vec{r}$ and $\vec{r}\,'$.

(d) Find the total charge induced on the conducting sphere. In addition, find the induced surface charge density on the sphere as a function of the polar angle θ.

III. Mathematical Physics

It is the hottest time of day in the Australian outback. Kelly the Komodo Dragon is tired, but realizes that if she sits down, the hot sand will burn tender parts of her skin. However, she reasons as follows.

Suppose it is possible to treat the sand as a continuous medium, with thermal conductivity κ, mass density ρ and heat capacity per unit mass s. The surface of the sand can be taken as the plane $z = 0$, and the sand obeys the heat conduction equation

$$\frac{\partial T}{\partial t} = \frac{\kappa}{\rho\, s} \frac{\partial^2 T}{\partial z^2} \, ,$$

where here T is the *difference* between the actual temperature and the time averaged temperature, t is the time of day and z the distance below the surface. The surface temperature at the surface $z = 0$ is given by

$$T = T_0 \, \cos \left[2\pi \, \frac{t - t_0}{t_d} \right] ,$$

with T_0 a constant with units of temperature, $t_d = 24 \; hours$, and t_0 is the time of the hottest part of the day.

Kelly then wonders if it would be more comfortable to use her tail to flip away a suitably thick top layer of sand before sitting down.

(a) Think of the earth to be infinite in extent in the $+z$ direction. (Recall that we defined the positive z direction to point downward into the earth from the surface.) Use separation of variables, together with the given boundary conditions, to find how the temperature inside the earth varies with z when $t = t_0$.

(b) Use this result to determine the thickness of the layer of sand that must be removed for Kelly to have the coolest surface to sit on. You may take $s = 800 \; J \; kg^{-1} K^{-1}$, $\rho = 1600 \; kg \; m^{-3}$, $\kappa = 0.3 \; J \; sec^{-1} m^{-1} K^{-1}$. You may also assume that the time-averaged temperature is independent of z.