Graduate Qualifying Exam, August'06

DAY TWO – Modern Physics

August 23, 2006

Please read carefully before starting –

On this second day of the exam we ask you to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

Note that you are expected to work all the problems covered in the exam.

Physical constants (which you may find useful although you may not need all of them)

$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N\,m^2}$	$\mu_0 = 4\,\pi \times 10^{-7}\ \mathrm{N\,s^2/C^2}$
$c = 1/\sqrt{\epsilon_0\mu_0} = 3.0 \times 10^8\ \mathrm{m/s}$	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
$h = 6.626 \times 10^{-34}\ \mathrm{J\,s}$	$\hbar = h/2\pi = 6.582 \times 10^{-22}\ \mathrm{MeV\,s}$
$\hbar c = 200\ \mathrm{eV\,nm}$	$(\hbar c)^2 = 0.389\ \mathrm{GeV^2\,mbarn}$
$m_e = 0.511\ \mathrm{MeV}/c^2$	$m_p = 938.3\ \mathrm{MeV}/c^2 = 0.9998\ amu$
$m_\mu = 105.7\ \mathrm{MeV}/c^2$	$m_{\pi^0} = 135\ \mathrm{MeV}/c^2$
$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$	$1\ \mathrm{eV}/c^2 = 1.783 \times 10^{-36}\ \mathrm{kg}$
$k_B = 1.381 \times 10^{-23}\ \mathrm{J/K}$	$N_A = 6.022 \times 10^{23}\ \mathrm{mol^{-1}}$
$1\ \mathrm{Mpc} = 3.086 \times 10^{22}\ \mathrm{m}$	$G_N = 6.674 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$

IV. Quantum Mechanics

A Stern-Gerlach experiment that analyzes spin along the i-direction (SG; i) is characterized by the Hamiltonian $H_i = \mu B S_i$, where S_i is the i-th component of the $\vec{S}$ spin operator.

(a) Consider spin-1 particles and use the following basis of S_z eigenstates: $\{|1\rangle, |0\rangle, |-1\rangle\}$. What eigenvalue(s) does $\vec{S}^2$ have for these states?

(b) Using the S_i commutation relations, derive raising and lowering operators, $S_\pm$, which obey the commutation relations $[S_z, S_\pm] = \pm\hbar S_\pm$, $[S_+, S_-] = 2\hbar S_z$, and $S_+ = S_-^\dagger$.

(c) Assuming m takes on a value such that

$$S_\pm \mid m\rangle \neq 0, \tag{1}$$

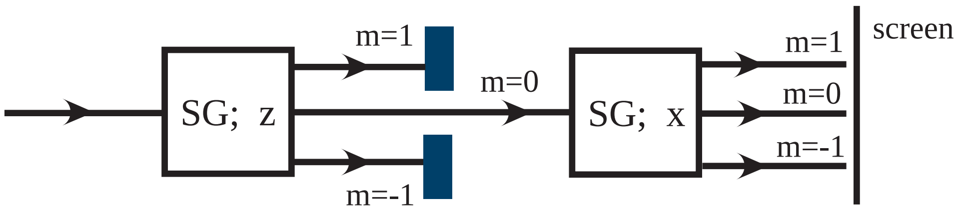
show that

$$S_\pm \mid m\rangle = N_m \mid m \pm 1\rangle \tag{2}$$

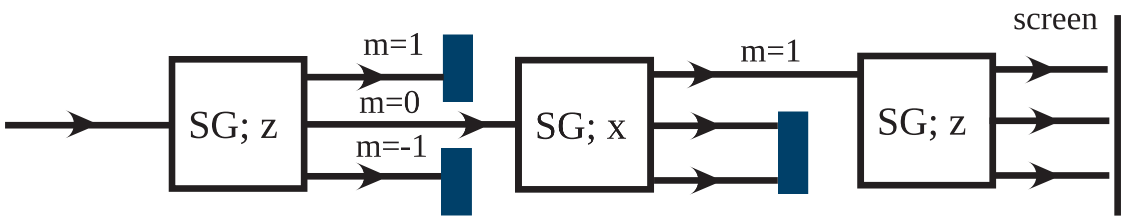
and determine the proportionality constant N_m .

(d) What is the 3×3 matrix for S_x in this basis?

(e) A beam of spin-1 particles, with random spin orientations, enters a (SG; z) apparatus. The $m = \pm 1$ states are blocked, with $m = 0$ transmitted to a second (SG; x) apparatus. What are the relative intensities for the beams measured at the screen? If you were unable to do the previous problem then explain how you would go about solving this problem. Using bra and ket notation is suggested.



(f) Now suppose that the $m = +1$ state is transmitted to another (SG; $\hat{z}$) apparatus as shown below. What is the relative intensity of the beams measured in this case?



V. Statistical Mechanics and Thermodynamics

Consider the following very simple model of a rubber band. Describe the rubber band as a one-dimensional polymer, with N monomers of length d that can point in either the $+z$ or $-z$ direction. One end of the polymer is attached to a point a couple of meters above the floor, and the other is attached to a mass M. The whole thing is in thermal equilibrium at temperature T.

(a) Calculate the partition function.

(b) Calculate the free energy.

(c) Calculate the energy.

(d) Calculate the average length $\langle L \rangle$ of the polymer.

(e) Based on the results you've obtained for this simple model of a rubber band, show whether the mass rises or falls when the temperature is increased.

(f) If you could obtain a real rubber band, you would be able to confirm that it gets hotter when stretched, and cooler when released. Therefore, for real rubber bands, $\left(\frac{\partial T}{\partial L}\right)_S > 0$, where S is the entropy, as usual.
From this result and your knowledge of how to derive thermodynamic identities, determine whether a mass hanging from the end of a real rubber band, as in part (e), should rise or fall when the rubber band is heated.

VI. General Physics (Note: parts (a),(b),(c),(d) are not related to each other.)

(a) (i) The intensity of electromagnetic radiation from the Sun arriving at the Earth, is 1.4 kW/m^2. Compute the radiation pressure on a "solar sail" placed at a distance from the Sun equal to the Earth's orbit radius.
(ii) The cloud of particles known as "solar wind" which is continually ejected from the Sun, has an average density of 7 protons per cubic centimeter and moves at an average speed of 400 km/s. Compute the pressure on the solar sail due to that solar wind.

(b) The famous supernova SN1987A occurred at a distance of about 5×10^{12} *light-seconds* from Earth. For the first time in history this supernova was also observed via the burst of neutrinos it emitted. Within that burst, one neutrino of 20 MeV energy and another neutrino of 10 MeV energy arrived on Earth within ten seconds of each other. Assuming that both were emitted simultaneously at the same location, derive an approximate upper limit on the neutrino mass, in units of eV/c^2.

(c) One of the mechanisms of energy production in stars is the "proton-proton cycle" which eventually fuses four protons into an alpha particle:

– proton-proton fusion, $\quad {}^1H + {}^1H \to {}^2H + e^+ + \nu_e + \gamma$ (deuterium formation)
– deuterium-proton fusion, $\quad {}^2H + {}^1H \to {}^3He + \gamma$
– helium-3 fusion, $\quad {}^3He + {}^3He \to {}^4He + {}^1H + {}^1H$

The masses of hydrogen, deuterium, helium-3, and helium-4 are, respectively, $m({}^1H) = 1.008\ amu$, $m({}^2H) = 2.014\ amu$, $m({}^3He) = 3.016\ amu$, and $m({}^4He) = 4.003\ amu$.
(i) Estimate the total amount of energy in MeV released in one cycle.
(ii) Which of the above reactions would you expect to have the slowest rate, and why? Can you think of an important consequence of this on the evolution of the solar system?

(d) Astronomical data are often based on absorption lines seen in light originating in stars but passing through interstellar material. Consider the simple case of a cloud of (cold) atomic hydrogen illuminated by a star "behind" it whose spectrum has uniform intensity from 1000 to 10000 *Angstroms* and is zero at all other wavelengths. For parts (i) and (ii) assume the cloud to be at rest relative to the star.
(i) What absorption lines can be observed? Give wavelengths and quantum number changes.
(ii) Studying the lowest energy absorption line, an observer with a high-resolution spectrometer notices that the line is split into a series of closely spaced lines. Assuming that this is due to the cloud being situated in a weak magnetic field, write down the perturbing Hamiltonian that gives rise to the splitting. Derive a "calibration" formula which gives the magnetic field in terms of the smallest observed splitting, $\Delta\lambda$. Sketch a level diagram and label the quantum numbers.
(iii) Actually, the gas cloud may be moving relative to the star. What is the relative velocity between the cloud and the star if the lowest energy absorption line is red-shifted by 14% of the wavelength given off by the star?