Graduate Qualifying Exam, August'06

DAY ONE – Classical Physics

August 22, 2006

Please read carefully before starting –

On this first day of this exam you are asked to work three problems, each of which has several parts.

Work each problem in a separate blue exam book. Write your name and the problem number on the front cover of each.

In order to get full credit you must show all your work, either by showing all relevant steps of a calculation or, where applicable, by giving a clear and logically consistent explanation. Correct answers with no supporting calculation or explanation will receive little or no credit. In case of an incorrect final answer, partial credit will be given if a correct approach to the problem is evident.

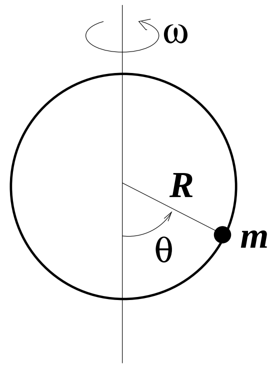
Note that you are expected to work all the problems covered in the exam.

Physical constants (which you may find useful although you may not need all of them)

$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N\,m^2}$	$\mu_0 = 4\,\pi \times 10^{-7}\ \mathrm{N\,s^2/C^2}$
$c = 1/\sqrt{\epsilon_0\mu_0} = 3.0 \times 10^{8}\ \mathrm{m/s}$	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
$h = 6.626 \times 10^{-34}\ \mathrm{J\,s}$	$\hbar = h/2\pi = 6.582 \times 10^{-22}\ \mathrm{MeV\,s}$
$\hbar c = 200\ \mathrm{MeV\,fm}$	$(\hbar c)^2 = 0.389\ \mathrm{GeV^2\,mbarn}$
$m_e = 0.511\ \mathrm{MeV}/c^2$	$m_p = 938.3\ \mathrm{MeV}/c^2$
$m_\mu = 105.7\ \mathrm{MeV}/c^2$	$m_{\pi^0} = 135\ \mathrm{MeV}/c^2$
$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$	$1\ \mathrm{eV}/c^2 = 1.783 \times 10^{-36}\ \mathrm{kg}$
$k_B = 1.381 \times 10^{-23}\ \mathrm{J/K}$	$N_A = 6.022 \times 10^{23}\ \mathrm{mol^{-1}}$
$1\ \mathrm{Mpc} = 3.086 \times 10^{22}\ \mathrm{m}$	$G_N = 6.674 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$

I. Classical Mechanics

A small bead of mass m is threaded on a frictionless circular wire hoop of radius R and negligible mass. The hoop lies in a vertical plane, which is forced to rotate about the hoop's vertical diameter with constant angular speed ω. The bead's position on the hoop is specified by the angle θ measured up from the bottom of the hoop. Let g denote the vertical acceleration due to gravity at the Earth's surface, as usual.



(a) Write down the Lagrangian $L(\theta, \dot{\theta})$ of the system in terms of the generalized coordinate θ and its time derivative $\dot{\theta}$.

(b) Show that the equation of motion for the bead is

$$\ddot{\theta} = (\omega^2 \cos\theta - g/R) \sin\theta.$$

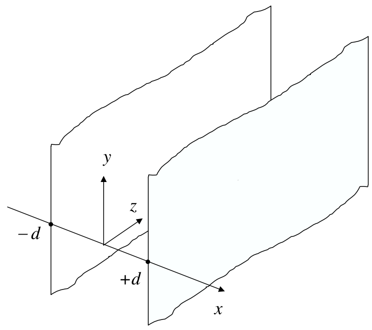
(c) When the angular speed ω is smaller than some critical speed ω_c, there are only two equilibrium positions for the bead, but for speeds $\omega > \omega_c$, there are four equilibrium positions. Determine ω_c, the two equilibrium positions for $\omega < \omega_c$, and the four equilibrium positions for $\omega > \omega_c$. Carefully explain your arguments.

(d) For all of the equilibrium positions found above, determine if each is a stable or an unstable equilibrium. Again, carefully explain your reasoning.

(e) Let θ_0 denote one of the stable equilibrium positions for $\omega > \omega_c$. Find the frequency of small oscillation of the bead about θ_0.

II. Electromagnetism

A uniform sheet of dielectric material extends from $-d$ to $+d$ in the x-direction and to $\pm\infty$ in the y and z directions. The sheet has a dielectric constant $\epsilon > 1$, is nonmagnetic, and is surrounded by air ($\epsilon = 1$). Consider an electromagnetic wave propagating in the z-direction through the sheet. Assume that the mode under consideration has $B_x = B_z = 0$, and that the fields do not vary in the y direction. Below it will become apparent that the structure *guides* the wave, in the sense that the fields decrease rapidly for $|x|$ outside the sheet.



(a) Give the set of Maxwell's equations appropriate to this situation, and deduce the wave equations for B_y both inside and outside the sheet. Indicate which components of $\vec{E}$ will be nonzero.

(b) With the ansatz that $\vec{E}$ and $\vec{B}$ have given frequency ω and wavenumber k in the z-direction, deduce the partial differential equations determining the x-dependence of B_y in each region and give expressions for the other field components in terms of B_y.

(c) What conditions must the fields satisfy at $x = d$?

(d) Show that there are solutions for B_y which: (i) are even functions of x, (ii) satisfy the boundary conditions at $x = d$, and (iii) rapidly decay for $x > d$, provided that $kc/\epsilon^{1/2} < \omega < kc$ and the dispersion relation

$$\epsilon\beta = \alpha \tan(\alpha d) \tag{1}$$

holds, where

$$\alpha = \sqrt{\epsilon\omega^2/c^2 - k^2} \tag{2}$$

$$\beta = \sqrt{k^2 - \omega^2/c^2}. \tag{3}$$

[NOTE: The formulas are given in Gaussian units, but you may use any consistent set of units.]

III. Mathematical Physics

The diffusion of neutrons through a block of U^{235} is approximately described by the diffusion equation

$$\frac{\partial n(\vec{x},t)}{\partial t} = D\nabla^2 n(\vec{x},t) + \lambda n(\vec{x},t),$$

where $n(\vec{x},t)$ is the neutron number density at point $\vec{x}$ and at time t. $D\,(>0)$ is the diffusion coefficient and the term $\lambda n(\vec{x},t)$ with $\lambda > 0$ describes the fact that fission adds neutrons to the system at a rate proportional to the number density.

(a) Find the general solution (that is, find $n(\vec{x},t)$ in terms of $n(\vec{x},0)$) for a cubic block of U^{235} of length L subject to the boundary condition that $n(\vec{x},0) = 0$ for $\vec{x}$ on the surface of the cube.

(b) Discuss what is meant by the "critical size" of the block and show that the critical size is of the form

$$L_c = \sqrt{\frac{3\pi^2 D}{\lambda}}.$$

(c) In order to use the above result, suppose we instantaneously assemble eight barely subcritical cubes into a single cube. Find the time constant of the resulting catastrophe.