Week 6

Combining Measurements

Goodness of Fits: some examples

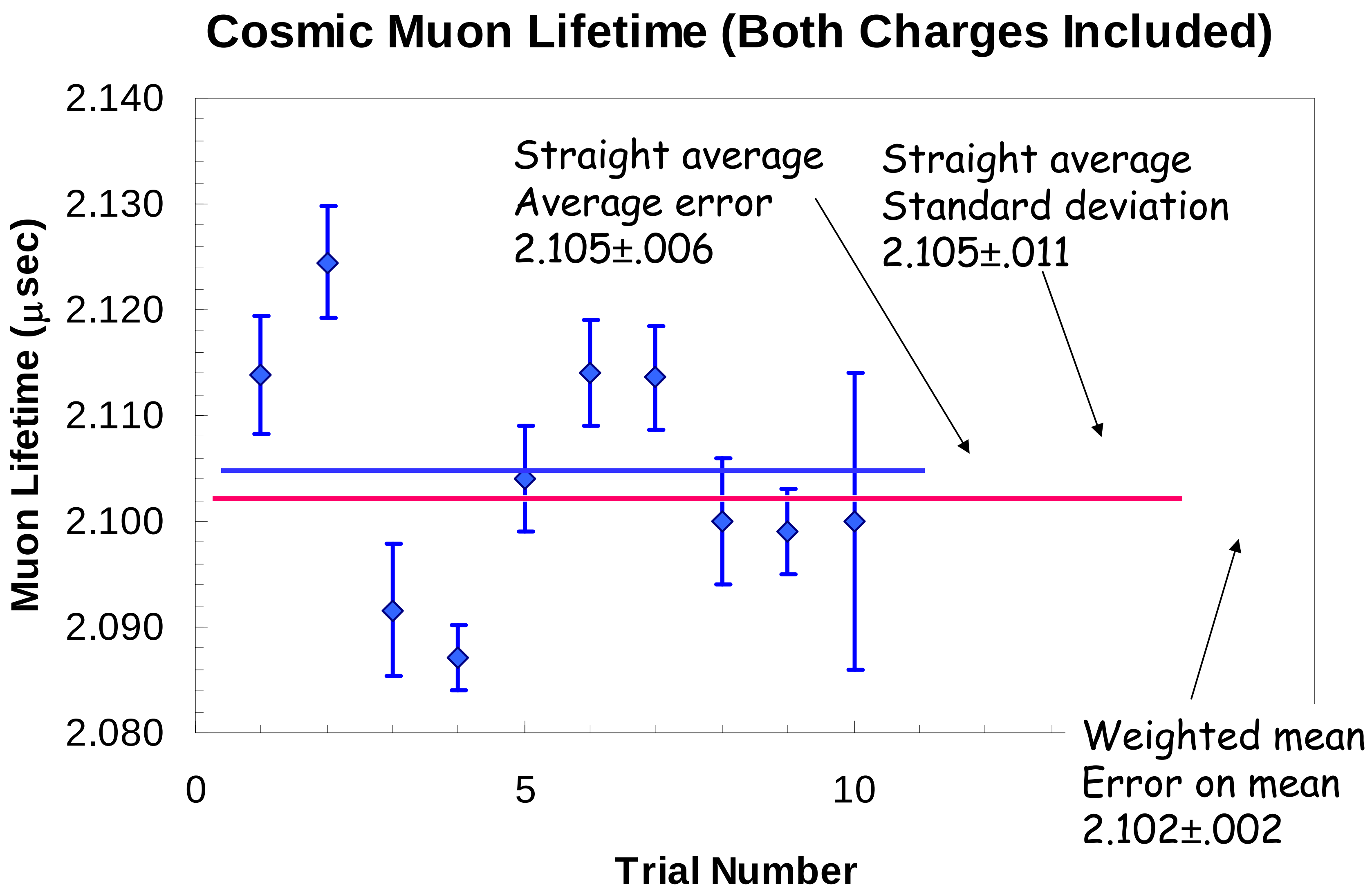
Best way to Combine Numbers

- Suppose we have N measurements of the same physical quantity. How best to combine?
- Must test agreement or internal consistency
- Plot the data to get a visual sense for the numbers!
- cf. Bevington p. 56 ff.

Actual Muon Lifetime data from MPL:

Trial	Lifetime	Error
	μsec	μsec
1	2.1138	0.0056
2	2.1245	0.0053
3	2.0916	0.0062
4	2.0871	0.0010
5	2.1040	0.0050
6	2.1140	0.0050
7	2.1136	0.0049
8	2.1000	0.0060
9	2.0990	0.0040
10	2.1000	0.0140

These numbers are not consistent…so what to do?



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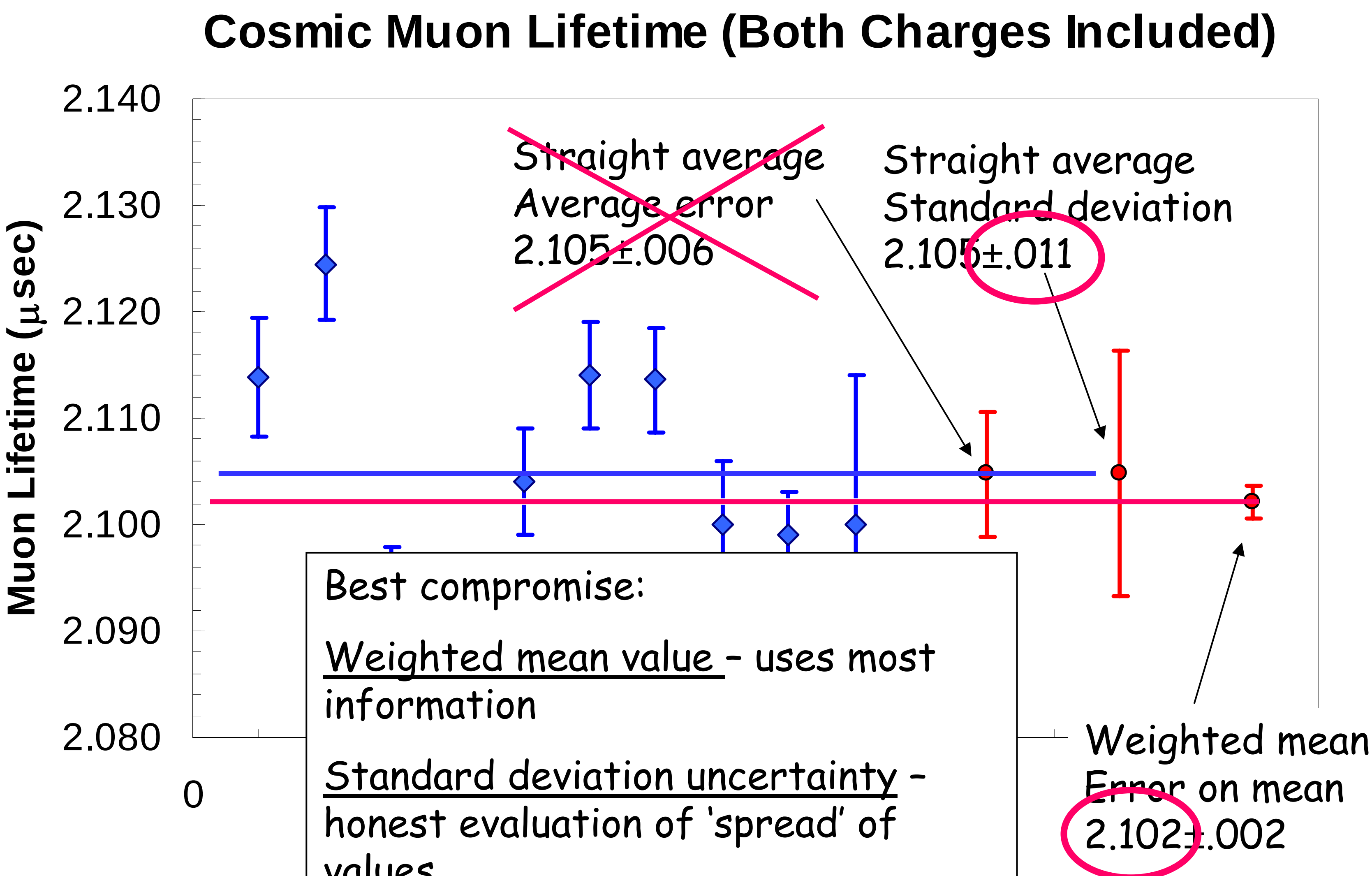


Best compromise:

Weighted mean value – uses most information

Standard deviation uncertainty – honest evaluation of 'spread' of values

Umm... weighted mean?

$$P(\overline{\tau}) = \Pi_{i=1}^{N}\left\{\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_i}e^{-\frac{1}{2}\left(\frac{\tau_i-\overline{\tau}}{\sigma_i}\right)^2}\right\} = \left\{\Pi_{i=1}^{N}\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_i}\right\}e^{-\frac{1}{2}\sum_{1}^{N}\left(\frac{\tau_i-\overline{\tau}}{\sigma_i}\right)^2}$$

Maximum likelihood function maximizing P and maximizing ln P amount to the same thing

Assume each data point has a Gaussian error and that all measurements are independent

$$\frac{d}{d\overline{\tau}}\left(-\frac{1}{2}\sum_{i=1}^{N}\left(\frac{\tau_i-\overline{\tau}}{\sigma_i}\right)^2\right) = -\sum_{i=1}^{N}\frac{\tau_i-\overline{\tau}}{\sigma_i^2} = 0$$

Solve for best value of τ

$$\overline{\tau} = \sum_{i=1}^{N}\frac{\tau_i}{\sigma_i^2}\Bigg/\sum_{j=1}^{N}\frac{1}{\sigma_j^2}$$

Weighted Mean

two separate sums

Weighted Mean

$$\bar{\tau} = \sum_{i=1}^{N} \frac{\tau_i}{\sigma_i^2} \Big/ \sum_{j=1}^{N} \frac{1}{\sigma_j^2}$$

Weighted Mean

$$\sigma_{\bar{\tau}}^2 = \sum_{i=1}^{N} \sigma_i^2 \left(\frac{\partial \bar{\tau}}{\partial \tau_i} \right)^2 + \cancel{correlations}$$

How much does the answer change if the i^{th} data point changes by a little bit?

$$\frac{1}{\sigma_{\bar{\tau}}^2} = \sum_{i=1}^{N} \frac{1}{\sigma_i^2}$$

Uncertainty on the Weighted Mean

mnemonic formula

Why is my χ^2 so bad? Case 1

Why is my χ^2 so bad? Case 1

R.A.Sch. v.9

Date Now: Time Now: Log Book Page:

	Value	Initial Step Size	Final Step Size	Estimated Uncertainty	Refined Uncertainty
Fit Parameter 1=	-0.24	0.1	0.0063		
Fit Parameter 2=	1.05	0.1	0.0016		
Fit Parameter 3=					
Fit Parameter 4=					
Fit Parameter 5=					
χ^2 =	55.78				
χ^2/d.o.f. =	3.10				

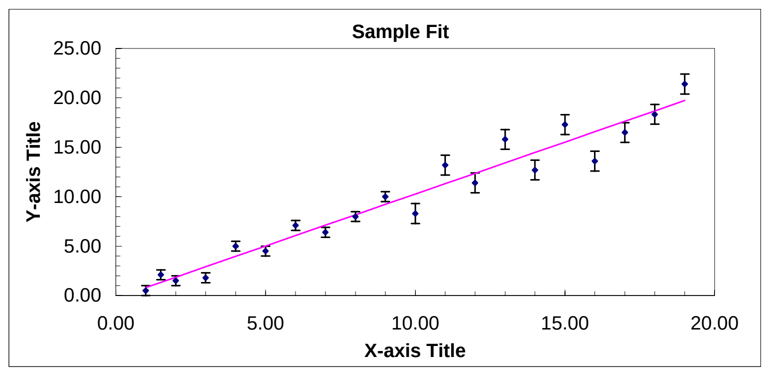
Y = FP1 + FP2 X

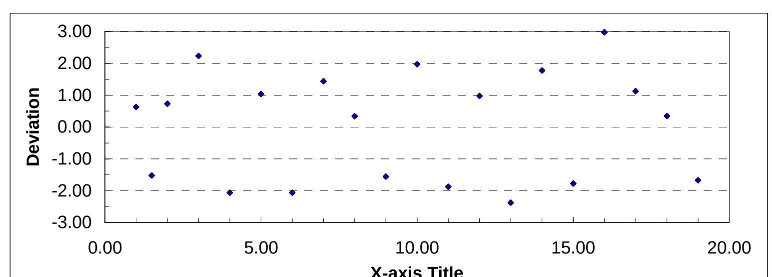
Entries	Constraints	d.o.f.	$\sigma(\chi^2$/dof)
20	2	18	0.33

Substitute your data (up to 1090 lines) below. Program "FIT Theory" column with your fitting function.

	X Data	Y Data	DY Data	FIT Theory	Deviation	Dev. ^{2}
1	1.00	0.50	0.50	0.81	0.63	0.40
2	1.50	2.10	0.50	1.34	-1.52	2.31
3	2.00	1.50	0.50	1.87	0.73	0.53
4	3.00	1.80	0.50	2.92	2.23	4.98
5	4.00	5.00	0.50	3.97	-2.07	4.27
6	5.00	4.50	0.50	5.02	1.04	1.07
7	6.00	7.10	0.50	6.07	-2.06	4.26
8	7.00	6.40	0.50	7.12	1.44	2.07
9	8.00	8.00	0.50	8.17	0.34	0.12
10	9.00	10.00	0.50	9.22	-1.56	2.43
11	10.00	8.30	1.00	10.27	1.97	3.89
12	11.00	13.20	1.00	11.32	-1.88	3.53
13	12.00	11.40	1.00	12.37	0.97	0.95
14	13.00	15.80	1.00	13.42	-2.38	5.65
				7	1.77	3.15
				3	-1.77	3.15
				3	2.98	8.86
				3	1.13	1.27
19	18.00	18.33	1.00	18.68	0.35	0.12
20	19.00	21.40	1.00	19.73	-1.67	2.79

$\Delta_i > \sigma_i$ leads to $\chi^2/\nu > 1$





χ^2 Tolerance = +- 0.010

Fit may be OK but the errors are underestimated

You may not inflate the σ_i's to get to $\chi^2/\nu \sim 1$

Why is my χ^2 so bad? Case 2

Why is my χ^2 so bad? Case 2

R.A.Sch. v.9

Date Now: Time Now: Log Book Page:

$$Y = FP1 + FP2\ X$$

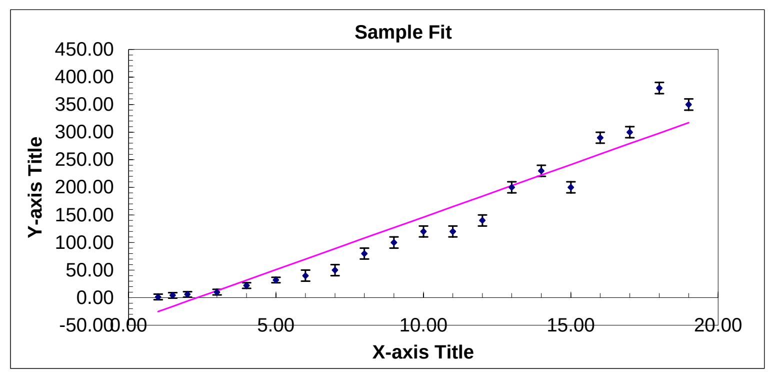
Entries	Constraints	d.o.f.	$\sigma(\chi^2/dof)$
20	2	18	0.33

	Value	Initial Step Size	Final Step Size	Estimated Uncertainty	Refined Uncertainty
Fit Parameter 1=	-44.36	1.0	0.1250		
Fit Parameter 2=	19.04	1.0	0.0625		
Fit Parameter 3=					
Fit Parameter 4=					
Fit Parameter 5=					
χ^2 =	262.58				
χ^2/d.o.f. =	14.59				

tting function.

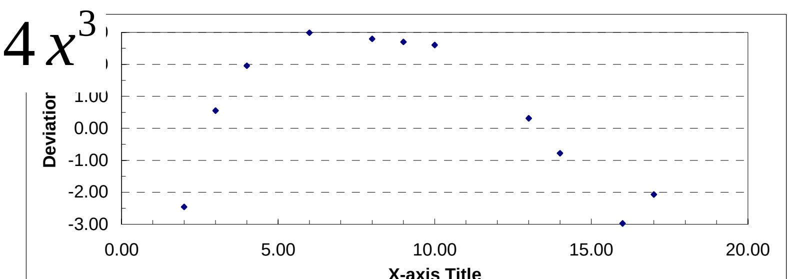
Errors may be estimated correctly but the parent distribution from which data are drawn is not correct

						ev. ^{2}
						3.36
						5.01
						.04
						.30
						.84
						4.18
						.92
						5.14
						.81
10	9.00	100.00	10.00	126.99	2.70	7.28
						6.77
						20.31
						19.45
						0.10
						0.61
						16.99
						8.84
						4.28
						66.69
						10.64



χ^2 statistic "can't tell" if the data is bad or the fit function is bad.

$$Y = FP1 + FP2\,x + FP3\,x^2 + FP4\,x^3$$



Adding terms will definitely reduce χ^2/ν, but caution: the fit values may be nonsense

χ^2 Tolerance = +- 0.010

Why is my χ^2 so bad? Case 2

$$y_{fit} = a_1 + a_2 x + \underbrace{a_3 x^2 + a_4 x^3}_{\text{VALID NEW TERMS?}}$$

Is the change in χ^2 enough better to justify extra terms but not "too good to be true"?

$$F_x = \frac{\chi^2(n=2) - \chi^2(n=3)}{\chi^2_\nu(n=3)}$$

The "F-test" statistic

see Bevington p.207

Ask: is $\sigma_{a3} > |a_3|$? If so, the added term may not be meaningful.

Also, if n → N, we can get χ^2 → 0, which is nonsense

Why is my χ^2 so bad? Case 3

Why is my χ^2 so bad? Case 3

R.A.Sch. v.9

Date Now: Time Now: Log Book Page: ________

$$Y = FP1 + FP2\ X + FP3\ X^2$$

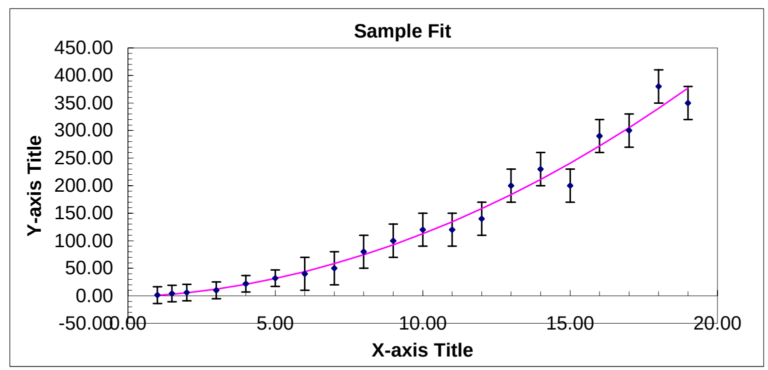
	Value	Initial Step Size	Final Step Size	Estimated Uncertainty	Refined Uncertainty
Fit Parameter 1=	-2.29	0.1	0.0500		
Fit Parameter 2=	2.06	0.1	0.0500		
Fit Parameter 3=	0.94	0.1	0.0031		
Fit Parameter 4=					
Fit Parameter 5=					
χ^2 =	6.38				
χ^2/d.o.f. =	0.38				

Entries	Constraints	d.o.f.	$\sigma(\chi^2/dof)$
20	3	17	0.34

Substitute your data (up to 1090 lines) below. Program "FIT Theory" column with your fitting function.

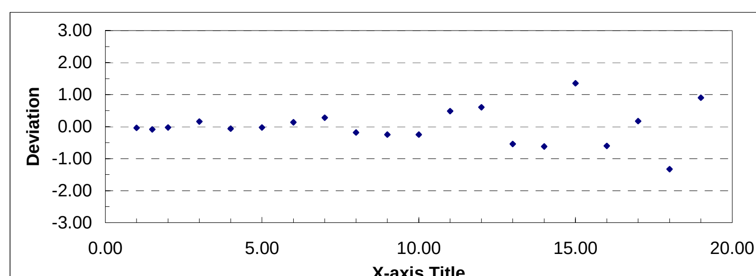
	X Data	Y Data	DY Data	FIT Theory	Deviation	Dev. ^{2}
1	1.00	1.30	15.00	0.72	-0.04	0.00
2	1.50	4.20	15.00	2.93	-0.08	0.01
3	2.00	6.00	15.00	5.61	-0.03	0.00
4	3.00	10.00	15.00	12.39	0.16	0.03
5	4.00	22.00	15.00	21.05	-0.06	0.00
6	5.00	32.00	15.00	31.59	-0.03	0.00
7	6.00	40.00	30.00	44.02	0.13	0.02
8	7.00	50.00	30.00	58.33	0.28	0.08
9	8.00	80.00	30.00	74.53	-0.18	0.03
10	9.00	100.00	30.00	92.61	-0.25	0.06
11	10.00	120.00	30.00	112.58	-0.25	0.06
12	11.00	120.00	30.00	134.43	0.48	0.23
13	12.00	140.00	30.00	158.16	0.61	0.37
14	13.00	200.00	30.00	183.78	-0.54	0.29
				.28	-0.62	0.39
				.67	1.36	1.84
				.94	-0.60	0.36
				.09	0.17	0.03
19	18.00	380.00	30.00	340.13	-1.33	1.77
20	19.00	350.00	30.00	377.06	0.90	0.81

$\Delta_i < \sigma_i$ leads to $\chi^2/\nu < 1$



Fit is "no better" than a fit with $\chi^2/\nu \sim 1$

σ_i's have been overestimated: the data is better than you think



χ^2 Tolerance = +- 0.010

Why is my χ^2 so bad? Case 4

Maybe you are just unlucky.

How close to "1" does χ^2 have to be?

Without proof we give:
(Ambramowicz & Stegun p.943)

$$\sigma_{\chi^2_\nu} = \sqrt{2/\nu}$$

Thus, for a "good" fit:

$$\chi^2_\nu = 1 \pm \sqrt{2/\nu}$$

e.g. for N=200 (x_i, y_i, σ_i) points fit to a straight line n=2, then is "good"

$$\chi^2_\nu = 1 \pm \sqrt{2/198} = 1 \pm 0.10$$

Summary

- Combine numbers using the weighted mean, but be cautious about assigning a fair uncertainty
- Memorize the weighted mean formulas
- The χ^2 statistic can't distinguish between you messing up the errors or picking a bad parent function: You've got to figure it out.
- χ^2 is itself a statistical quantity, sometimes even a "good" fit has a bad χ^2 (but don't bet your career on it).

Backup Slides…