Week 2(b)

Bevington & Robinson Essentials: review/learn Chapters 1, 2, 3

Chapter 1: Uncertainties
Chapter 2: Probability Distributions
Chapter 3: Error Propagation
Four Important Probability Distributions

1. Binomial
   - heads / tails on coin flips
   - left / right-handedness in people

2. Poisson
   - “counting” experiments with random uncorrelated “events”
   - radioactive decay statistics

3. Gaussian
   - model for random fluctuations in experimental data

4. Lorentzian
   - resonance response in mechanical and quantum systems
Binomial Distribution

- draw colored balls from an “urn”, say red and blue
- draw a ball, record color, replace ball so red/blue ratio never changes

Let \( p = \) probability of drawing a red ball, e.g. 1/5
Let \( q = \) probability of drawing a blue ball, e.g. 4/5

\[ p + q = 1 \text{ : sum of probabilities must be “100%”} \]

Probability of drawing 3 red in a row and then 2 blue in a row (5 total balls)

\[
P(r,r,r,b,b) = \frac{1}{5_1} \times \frac{1}{5_2} \times \frac{1}{5_3} \times \frac{4}{5_4} \times \frac{4}{5_5} = 0.0051 = p^3q^2
\]

If we don’t care about what order we select the 3 red and 2 blue, we can permute the ordering to find all possible combinations of drawings

\[
\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1) \times (2 \cdot 1)} = \frac{5!}{3!2!} = 10 = \text{ways of permuting 5 items} \div \text{ways of permuting 3 items} \times \text{ways of permuting 2 items}
\]
\[ P(x = 3 \text{ red}; n = \text{ red + blue} = 5, p(\text{red}) = 1/5) = \frac{n!}{x!(n-x)!} p^x q^{n-x} = 10(0.0051) \rightarrow 5\% \]

Generalize to the Binomial Distribution:

\[ P_{\text{Binomial}}(x; n, p) = \frac{n!}{x!(n-x)!} p^x q^{n-x} \]

- defined for integer \( x \) values
- \( n \), the number of “trials”, must be small enough to compute the factorial
- \( p \) represents “success”
- \( q \) represents “failure”

Example: throw 10 dice at once. What is the probability that 3 of them show a “6”?

\[ P_{\text{Binomial}}(3; 10, \frac{1}{6}) = \frac{10!}{3!7!} \left( \frac{1}{6} \right)^3 \left( \frac{5}{6} \right)^{10-3} = 7 \]

\[ = 120(0.0046)(0.2791) \rightarrow 15.5\% \]
What is the average or mean value of $x$? Call it $\mu$.

E.g. what is the mean number of red balls out of 5 drawn, or the mean number of “6”'s among 10 dice thrown?

First note the “binomial theorem” of algebra:

$$(p + q)^n = \sum_{x=0}^{n} \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

Since $p+q = 1$, the sum over all values of $x$ is unity. This is required for a proper probability distribution.

The mean value of anything is a probability-weighted sum, written as:

$$\mu = \sum_{x=0}^{n} x \cdot P(x; n, p)$$

We need to evaluate this sum, but the result is (Assignment problem 2.5):

$$\mu = np$$  Binomial mean (average)

E.g.  $\mu_{\text{red balls}} = 5(1/5) = 1.00$ - these averages are not integers

$\mu_{\text{6's on dice}} = 10(1/6) = 1.67$
What is the variance, or “second moment”, of $x$? Call it $\sigma^2$. This is related to the square of the standard deviation.

$$\sigma^2 = \sum_{x=0}^{n} (x - \mu)^2 \cdot P(x; n, p)$$

Solve using the same method as on the assigned problem to get:

$$\sigma^2 = np(1 - p)$$  

Binomial variance

E.g.  

$$\sigma_{\text{red balls}} = \sqrt{5 \left(\frac{1}{5}\right) \left(1 - \frac{1}{5}\right)} = 0.89$$  - these describe the “width” of the distributions

$$\sigma_{\text{6's on dice}} = \sqrt{10 \left(\frac{1}{6}\right) \left(1 - \frac{1}{6}\right)} = 1.18$$
Poisson Distribution

Take the limit of the Binomial distribution when the $p$ of “success” is very small, i.e. $\mu \ll n$.

For example: (random) cosmic ray arrival times

The maximum number of cosmic rays arriving in one time bin is huge ($\rightarrow$ infinity) but in fact the average value of $x$ is a small number: $\mu$. Let $\mu$ be the average count of cosmic rays per unit time.

What is $P_{\text{Poisson}}(x; \mu)$? Suppose the average rate is 2.5/sec.

$P_{\text{Poisson}}(0; 2.5) = ?$  $P_{\text{Poisson}}(1; 2.5) = ?$  $P_{\text{Poisson}}(2; 2.5) = ?$  etc.
Take the relevant limit of the Binomial distribution:

\[ P_{\text{Binomial}} (x; n, p) = \left[ \frac{1}{x! (n-x)!} \right] p^x (1-p)^{n-x} \]

\[ \frac{n!}{(n-x)!} = n(n-1)...(n-(x+1)) \approx n^x \quad \text{Since } x \text{ is } \ll n. \]

\[ (1-p)^{-x} = 1 + px \rightarrow 1 \quad \text{As } p \rightarrow 0 \]

\[ (1-p)^n = (1-p)^{\mu/p} = \left( (1-p)^{-p} \right)^{\mu} \rightarrow (1/e)^\mu = e^{-\mu} \quad \text{Look it up…} \]

This is the Poisson Distribution:

\[ P_{\text{Poisson}} (x; \mu) = \frac{\mu^x}{x!} e^{-\mu} \]

- defined for integer \( x \) values
- \( \mu \) represents mean number of counts per interval
Our example:

\[ P_{\text{Poisson}}(0; 2.5) = \ ? \quad P_{\text{Poisson}}(1; 2.5) = \ ? \quad P_{\text{Poisson}}(2; 2.5) = \ ? \quad \text{etc.} \]

\[ P_{\text{Poisson}}(0; 2.5) = \frac{2.5^0}{0!} e^{-2.5} \rightarrow 8\% \]

\[ P_{\text{Poisson}}(1; 2.5) = \frac{2.5^1}{1!} e^{-2.5} \rightarrow 20\% \]

\[ P_{\text{Poisson}}(2; 2.5) \rightarrow 26\% \]

\[ P_{\text{Poisson}}(3; 2.5) \rightarrow 21\% \]

\[ P_{\text{Poisson}}(> 0; 2.5) = 1 - P_{\text{Poisson}}(= 0; 2.5) \rightarrow 92\% \]

Most probable number of counts (2) is not the same as the mean (2.5)
Check that the sum over all values of $x$ gives 100% probability:

$$\sum_{x=0}^{\infty} P_{\text{Poisson}}(x; \mu) = \sum_{x=0}^{\infty} \left[ \frac{\mu^x}{x!} \right] e^{-\mu} = e^{+\mu} e^{-\mu} = 1$$

Yes!

Is it really true that $\mu$ is the mean of this distribution?

$$\langle x \rangle = \sum_{x=0}^{\infty} x P_{\text{Poisson}}(x; \mu) = \mu e^{-\mu} \sum_{x=1}^{\infty} \left[ \frac{\mu^{x-1}}{(x-1)!} \right] = \mu e^{-\mu} e^{+\mu} = \mu$$

Yes!

What is the “width” of the distribution, i.e. the square-root of the variance?

$$\sigma^2 = \sum_{x=0}^{\infty} (x - \mu)^2 P_{\text{Poisson}}(x; \mu) = \sum_{x=0}^{\infty} x^2 \frac{\mu^x}{x!} e^{-\mu} - \mu^2 = \mu$$

$$\sigma^2 = \mu \quad \text{or} \quad \sigma = \sqrt{\mu}$$

For the Poisson distribution, the mean and the variance are the same.
Gaussian Distribution

Adapt the Binomial distribution to the limit when \( n \) gets large and the number of “successes” \((x\ or\ p)\) becomes large, too, so that \( np \gg 1\).

- The algebra is straightforward but lengthy, so we skip it
- Also, we can adapt the Poisson distribution (see Assignment)

\[
P_{\text{Poisson}}(x; \mu) \quad \mu > 10 \quad \rightarrow \quad p_{\text{Gaussian}}(x; \mu, \sigma)
\]

The result is the familiar Gaussian distribution:

\[
p_{\text{Gaussian}}(x; \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]

\( x \) is now a continuous variable, so \( p_{\text{Gaussian}} \) is now a probability density function, not the “probability of (integer) \( x \)”

\[
P_{\text{Gaussian}}(x_1 < x < x_2; \mu, \sigma) = \int_{x_1}^{x_2} p_{\text{Gaussian}}(x; \mu, \sigma) \, dx
\]

Must integrate the “density” (little \( p \)) to get the “Probability” (big \( P \)).
\[ P_{\text{Gaussian}}(x; \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

\[ P_{\text{Gaussian}}(x_1 < x < x_2; \mu, \sigma) = \int_{x_1}^{x_2} P_{\text{Gaussian}}(x; \mu, \sigma) \, dx \]

\( x \) can be a real number, but for counting experiments, where \( x \) is an integer

\[ \sigma^2 = \mu \quad \text{or} \quad \sigma = \sqrt{\mu} \]

like for the Poisson distribution

\[ \text{FWHM} \equiv \Gamma = (2.354)\sigma \quad \text{(see Assignment)} \]

FWHM is not simply twice \( \sigma \)

For non-counting experiments, \( x \) can be any value. People assume/wish that their random or instrumental fluctuations vary in a Gaussian way. Sometimes it is even true... but often it is not. Be on guard to “non-Gaussian” fluctuations in real experiments.
Memorize two oft-used elementary numerical facts:

\[ P_G(\mu - \sigma < x < \mu + \sigma) = \int_{\mu-\sigma}^{\mu+\sigma} p_{\text{Gaussian}}(x; \mu, \sigma) \, dx = 0.68 \rightarrow 68\% \]

\[ P_G(\mu - 2\sigma < x < \mu + 2\sigma) = \int_{\mu-2\sigma}^{\mu+2\sigma} p_{\text{Gaussian}}(x; \mu, \sigma) \, dx = 0.95 \rightarrow 95\% \]
Lorentzian Distribution

Also known as the “Cauchy” or the “Breit-Wigner” distribution, in context.

NOT related to the other three, but is crucial in the physics of resonances and quantum mechanical state transitions.

\[ \omega_0 = \sqrt{\frac{g}{L}} \]

Heisenberg:

\[ \Gamma \tau = \hbar \]

\[ \Delta E = E_\gamma \]
\[ P_{\text{Lorentzian}}(x; \mu, \Gamma) = \frac{\Gamma}{2\pi} \frac{1}{(x - \mu)^2 + (\Gamma/2)^2} \]

-x is a continuous variable, so \( P_{\text{Lorentzian}} \) is a probability density function, not a “probability of \( x \)”

\[ P_{\text{Lorentzian}}(x_1 < x < x_2; \mu, \Gamma) = \int_{x_1}^{x_2} P_{\text{Lorentzian}}(x; \mu, \Gamma) \, dx \]

Must integrate the “density” (little \( p \)) to get the “Probability” (big \( P \)).

“Big tails” compared to Gaussian line shape.

The second moment diverges, so we use \( \Gamma = \text{FWHM} \) to describe the function, rather than \( \sigma \).
Summary

• Learn about these 4 distributions and try to spot where they apply to your experiments in Modern Physics Lab.

• Do assigned problems for next week after reading Bevington chapters 1, 2, 3.