EXPERIMENTAL CHAOS

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I. INTRODUCTION

The last ten years has seen a revitalization in the field of classical mechanics due in part to the growing interest in nonperiodic systems and the regions of chaotic motion associated with these nonlinear systems. This experiment is designed to provide the student with an introduction to some of the new ideas which have not only led to renewed interest in classical mechanics but which also appear in other fields of physics and mathematics. Before continuing, read the Scientific American articles by Crutchfield and Sanders as introductions to chaos and fractals.

II. THE DUFFING OSCILLATOR

The apparatus is intended to be an experimental realization of the forced Duffing oscillator. The Duffing oscillator consists of a particle moving with linear damping in a potential well with a symmetric double minimum. The potential is given by the following expression:

$$V(x)=ax^4/4 - bx^2/2. \quad (1)$$

When a>0 and b>0, there are minima at $x_m = \pm(b/a)^{1/2}$. Also, in this case, V(0) is a maximum. Another useful quantity to know is the angular frequency of very small oscillations about the minima. This quantity is given by the expression $\omega_o^2=2|b|$, provided we assume a unit mass and small damping. *Plot the potential qualitatively and then verify these statements by explicitly deriving each expression and include your calculations in your lab book. The driven Duffing oscillator is forced with a time-dependent force given by $F(t)=F_0 \cos(\omega t)$.*

One can visualize the motion of the system as occurring on a torus (doughnut) with the x vs. v plane at any instant being the cross-section of this torus. The long way around the torus is related to the time variable. Every time the trajectory goes around the torus, it cuts through the x-v plane at equivalent times t, t+T, t+2T, etc. where T is the period of the driving force. The resulting pattern of spots gives rise to the Poincaré map.

Now use the Maple worksheet duffing.mws to explore the dynamics of the Duffing oscillator. Read Appendix A and set the values of F_0 and ω so you simulate trajectories where the oscillator: (i) stays in one potential well in periodic motion, (ii) samples both wells with periodic motion, and (c) the oscillator exhibits chaotic motion. The program plots x(t) vs. v(t). Be sure you create Poincaré maps for each type of motion and calculate the fractal dimension of each type of motion using techniques discussed later in the handout. *In your lab book, summarize in your own words how the Poincaré map relates to the trajectory you see in the simulation for each motion. Also, on a diagram of the potential, show qualitatively where the energy of the oscillator must be to attain the three different motions.*

III. THE APPARATUS

The oscillator is a vibrating reed (actually a 6" steel ruler) which is clamped at one end but free at the other end. Near the fixed end, the reed is driven by a force generated by a small loudspeaker cone and transmitted through a wooden dowel. The region on the reed near the fixed end is instrumented with a pair of strain gauges that respond to the curvature of the reed thereby providing a signal proportional to the amplitude of deflection of the free end. The free end is acted on by two permanent magnets positioned such that the end of the reed is attracted towards one or the other of the magnets (since the reed is made of magnetic material). The magnets and their attraction for the end of the reed provide the force that is equivalent to the double minimum of the potential of the mathematical Duffing oscillator. Do not run the apparatus at large amplitudes of deflection for extended periods when not taking data since the repeated strain at the dowel-ruler contact eventually will break the epoxy glue joint at this point.

The magnets' separation is controlled by a micrometer located on the right-hand side of the setup; its reading will be denoted as M_s. The position of the magnet assembly as a whole is varied with a second micrometer located to the left and whose setting will be referred to as M_t. With the magnets well separated, the potential has only a single minimum at x=0, roughly halfway between the magnets; the unstable maximum of the effective potential at x=0 is achieved by bringing the magnets closer together. The symmetry of the potential is controlled with M_t. In your notebook, sketch the qualitative behavior of the potential as M_t is varied. Note how w_0 in each well varies as you change M_t. When concluding your work for the day, separate the magnets as much as possible and position them so that the end of the reed is midway between them. This will help prevent excessive permanent magnetization of the end of the steel ruler.

Figure 1 shows a block diagram of the apparatus. The strain gauge signal is both amplified and differentiated. The amplified signal is available at the top right set of banana plugs (labeled "Ch 1") on the electronics chassis while the differentiated signal is available from the lower right set of banana plugs (labeled "Ch 2") on the same chassis. The gain for these signals is adjustable through the "Gain" switch located at the top right comer. By connecting the amplitude signal to the x-axis of an oscilloscope and simultaneously connecting the differentiated signal to the y-axis of the oscilloscope, one can obtain a phase plane trajectory of the motion of the system.

The electronics chassis also generates the sinusoidal signal which drives the reed through the loudspeaker. The amplitude of the signal is adjusted by the "Gain" knob and the frequency can be controlled by the "Freq" knob. Both knobs are located towards the left of the chassis. The frequency of the driver is monitored by measuring its period with a time interval counter. The signal driving that counter is a 25-microsecond pulse that occurs once per cycle and is available at a BNC connector (labeled "computer") at the back of the chassis. A second BNC connector (labeled "z-axis") provides a blanking voltage which when fed into the oscilloscope's z-axis input (in the rear of the scope) blanks out the trace of the oscilloscope except during the 25-microsecond pulse. After appropriate adjustment of the oscilloscope intensity (and the room lights), one can see a spot on the oscilloscope once per driving period. This spot is related to the Poincaré section.

Another set of banana plugs in the front of the electronics chassis are labeled "Meter"; and when used in connection with the voltmeter located below the chassis, they allow one to monitor the amplitude of the sine wave driving the reed. The final set of banana plugs is labeled "Out" and connects to the input cable to the loudspeaker to drive the system.

A second task of the pulse which drives the time interval counter is to serve as a trigger signal for the A/D hardware associated with the computer system. The directory *mpl* on the hard disk (drive c:) contains data logging program. The program **datalog** inputs both the x and v data once per driving cycle in response to the trigger signal and displays this data as it is being input. The range of the input is from -1024 to +1023. If you see any numbers equal to the extremes as data is being logged, then the output gain switch at the electronics chassis is set too high. Any time you use the logging program **datalog,** you should immediately plot your data to make sure that it is reasonable

IV. EXPERIMENTS WITH THE DUFFING OSCILLATOR

Following the instructions in this section, you will symmetrize the potential of the apparatus so it approximates the Duffing Oscillator as closely as possible. After that, you'll investigate the F_0 - w plane to find chaotic regions and measure the fractal dimension related to these regions. You'll conclude this section by confronting the fact that the experimental system consists of a reed and not a simple mass placed into a potential well.

The ODE approximately governing the motion of this system can be expressed as follows:

$$d^2x/dt^2 = -a\, x^3 + b\, x - c\, dx/dt + F_0 \cos(\omega t) \qquad (2)$$

The driving force, F_0, can be read off the D.C. voltmeter in unrelated units. The driving frequency, w, can be determined from the time interval counter which reads out the period.

Symmetrizing the Potential Wells:

Before starting, make qualitative sketches in your notebook of how the actual potential of the apparatus will vary as you move M_t. Make at least 5 sketches, with the following conditions that will arise as you monotonically move M_t (with no driving force): (a) the reed can only reside in the right well, (b) the reed can be in the left well even though the lowest energy state is in the right well, (c) the symmetric condition, (d) the reed can be in the right well even though the left well is the lowest energy state is in right well, and (e) the reed can only be in the left well.

Next discuss in your notebook why making the resonant frequency in both wells equal is equivalent to making the potential symmetric.

To symmetrize the wells, set M_s=0.075. Start by obtaining a "bow-tie" orbit with inversion symmetry. Use the storage oscilloscope with its cursors to see what M_t gives you this orbit. This is your first approximation to a symmetric potential. Next, drop the "Gain" knob to zero thereby causing the reed to fall into one of the wells. Also, decrease the "Freq" knob to some nearly zero value. You'll now want to AC couple the x-axis of the oscilloscope, turn off the room lights, and adjust the oscilloscope intensity so that the dot representing the Poincaré section is visible. You should also have the D.C. voltmeter connected to the "Meter" banana plugs to monitor the driving amplitude.

Now, proceed as follows:

1). Turn the "Gain" knob to some minimal value (for starters, try 0.1 volt. As you become more comfortable with the procedure, you'll want to decrease this value in an effort to get closer to the well minimum). Increase the "Freq" knob slowly; you should see the orbit increase in size to a maximum amount and then quickly decrease as you continue to increase the "Freq" knob setting. The system is at resonance when the orbit is at its maximum extent on the oscilloscope. You can determine the frequency for the current configuration by reading off the value of the period on the time interval counter and converting. In practice, it's difficult to determine exactly the maximum extent of the orbit on the oscilloscope screen. To correctly determine resonance conditions, we proceed to the next step.

2). Once near the resonance frequency for a given well, you can focus in on the correct frequency by quickly dropping the "Gain" to zero and observing the path of the Poincaré dot. When the spot traverses a linear path, the oscillator frequency and the free decay oscillation frequency are equal and you're at resonance. If you are off of resonance, the dot will spiral into its final resting point. Repeat step 1) and this step until the dot decays along a linear path. Record the resonant frequency.

3). In all likelihood, after your first attempt in one of the wells, the potential will not be symmetric. Adjust the "Freq" and "Gain" knobs to drive the reed into oscillation between the wells and then cut the "Gain" to zero to drop the reed into the other well (you may have to try this several times). Again, decrease the frequency.

4). Repeat steps 1) and 2) for this second well using the same initial "Gain" value. You'll now have two values for the resonance frequency (one for each well). These values must agree to within 5% before proceeding. To obtain a symmetric well and therefore like frequency values, you most probably will have to adjust M_t and repeat steps 1) and 2) for each well while using step 3) to jump between wells. Something useful to do is to plot the values of the resonance frequency for each well against M_t and use this plot to estimate the correct M_t setting. Also, between each cycle of frequency measurements, you should decrease the initial "Gain" value (eventually ending up in the milliVolt region) to get closer to the bottom of the wells and to thereby maximize the symmetry of the potential.

5). Once you are satisfied that your potential is symmetric, record the value of the resonant frequency for the symmetric well potential. You should also observe the phase space orbit (as opposed to the Poincaré spot trajectory) as the system decays at the resonance frequency value and make a sketch of this orbit for yourself. Note in your lab book whether the oscillator is under-, over- or critically damped. Give the observation which tells you which damping is present. Furthermore, carefully record your value of M_t since finding a symmetric potential is a somewhat tedious procedure.

You now have a symmetric well potential which should be used for the remainder of the experiment. You may find that just forming the proper bowtie is good enough to set M_t.

Investigating F_0-ω space:

Once the magnets have been positioned so that the potential is symmetric, you can investigate the behavior of the system as a function of the driver amplitude F_0 and the driving frequency, ω. You can think of these variables as coordinates in a plane which will be divided into sections that are distinguished by whether the motion is periodic or chaotic. A complete dissection of the plane into chaotic and periodic regions can be thought of as a phase diagram of the system, in the sense of liquid, solid, and gaseous phases in thermodynamic systems. It would be too time-consuming to determine completely the boundaries in this plane for a range of values of M_s, but you can qualitatively investigate some of its character. For example, you ought to be able to find, for a chosen value of M_s, values of F_0 and ω for which the system is oscillating in subharmonics of ω. Subharmonics can be identified by the presence of multiple Poincaré spots on some closed (and usually complicated) orbit. Find a set or two of F_0-ω values which give such behavior and sketch the orbits.

As you search the F_0-ω plane, you will find a strong history dependence to the system behavior. This is not uncommon in nonlinear systems. Be sure you use a fixed procedure to approach any point on the plane (e.g., approach a point from high ω at fixed F_0 and moving ω slowly).

You should also be able to find a region of the F_0-ω plane for which chaotic motion is observed. Chaotic regions are actually islands in the F_0-ω plane. Pick two points in such a region. You can measure the fractal dimension of the "strange attractor" for each point by performing the following steps. Run the program **datalog.** You'll want to log in between 1000 and 4096 data points. With the system in a chaotic orbit, hit the "Return" key on the computer to begin logging in the data.

Once the data logging procedure is completed, use the Excel spreadsheet, **mpl_chaos_box-counting.xls** to determine the fractal dimension of the strange attractor (see appendix B). If the (x,v) data points fall at random any place in the (x,v) plane, one expects that the points came from a set of dimensionality two. Therefore, one way to characterize the chaotic behavior of a system is to use the dimension of the Poincaré map. Using **mpl_chaos_box-counting.xls**, determine a reasonable method for extracting the fractal dimension for the Poincaré map from your chaotic region.

Look at your Poincaré map. Some regions are filled with points but other regions have no points. Run the same experiment twice. In your lab book, summarize what is the same and what is different between the Poincaré maps, nominally taken on the same oscillator driven at the same F_0 and ω. Are the fractal dimensions of the two plots the same to within uncertainties? In fact, is the plot from the box counting algorithm even a straight line? If not, why not? Can you still extract a valid fractal dimension? How can you find an uncertainty in that fractal dimension since you do not have an uncertainty on the data points used to find the fractal dimension? Comment in your lab book how the presence of open spaces on the Poincaré map indicates that chaotic motion is not random and cannot have a fractal dimension of 2.

Limitations of the system:

This is where we confront the fact that the reed is not a simple mass. The reed has many degrees of freedom whereas the mass has just one. There are many normal modes of oscillation for the reed; the experiment tries to use only the lowest mode, but complications from the higher modes are unavoidable. The next highest mode is at a frequency of ~100Hz and corresponds to the reed vibrating with one node between the tip and the clamped end. This mode is indeed excited during the experiment, and since taking a time derivative is equivalent to multiplying the signal by frequency, the presence of this higher mode is evident in the velocity signal (despite the fact that the electronics includes a low-pass filter above 50 Hz).

A consequence of the higher modes of the reed is more interesting. If you vary the frequency of the chassis oscillator in a regular manner, say with an increment dω, and at each frequency find the amplitude F_0 at which the motion goes from periodic to chaotic, you will have a set of points on the F_0 - ω plane. It is common practice to sketch a smooth curve through this set of points. To a first approximation, you may try to draw a straight line between two neighboring points at frequency values ω_1 and ω_2 and expect that you could use such a construction to estimate the value of F_0 where the system goes chaotic at some frequency $\omega=\omega_1 + d\omega/2$, for instance. In fact, this is not generally the case; this chaotic-periodic boundary is quite complicated. Try taking some data points (around 5 or so) at a chosen increment dω and sketch the hypothesized boundary. Then go back to intermittent values of ω and find the values of F_0 in these cases. As noted, you should in general find that the boundary is not as smooth as originally projected. A thorough investigation of this boundary is tedious and involves more time than available, but the result is that the boundary is a fractal with dimension of ~1.25.

Some final experiments:

Now explore the chaotic motion of the system as you vary some variable which sets the potential or the measurement. You may chose to move M_s, M_t, the phase of the Poincaré map relative to the driving force, or the procedure you use to approach a chaotic point on the F_0-ω plane.

References

Chemikov, A. A. **Physics Today**. 4L, 27, 1988. This article on chaos is posted in the chaos lab.

Crutchfield, J. P. et al. **Scientific American.** 255, 46, 1986. This article is one of the handouts and deals with chaos.

Mandelbrot, B. 1982: **Fractals.** This book is in the E & S library, call number 513 M27a.

Sanders, L. M. **Scientific American.** 256, 94, 1987. This article is one of the handouts and deals with fractals.

Appendix A: Documentation for the *duffing.mws* Maple worksheet

1. Introduction

duffing.mws is a Maple worksheet that solves the Duffing differential equation. Its purpose is to provide means to compare the experimental results of the Chaos experiment to the theoretical solution of the Duffing equation, given by

$$d^2x/dt^2 = -a\,x^3 + b\,(x-x_{asym}) - c\,dx/dt + F_0\cos(\omega t) \qquad (3)$$

(the same as above, except here I include x_{asym}, a measure of the potential well's asymmetry, that is zero for a symmetric well). In order to solve this equation using Maple, it is transformed into the following system of equations:

$$dx(t)/dt = v(t) \qquad (4)$$

$$dv(t)/dt = -a\,x^3 + b\,(x-x_{asym}) - c\,dx/dt + F_0\cos(\omega t) \qquad (5)$$

2. Using duffing.mws

Open the worksheet.

Set a, b, c, xasym, ω, F0, x0, and v0 as desired.

Examples:
chaos: 1, 1, 1, 0, 1, 0.9, 0, 0 (respectively)
third subharmonic: 1, 1, 1, 0, 1, 0.85, 0, 0 (respectively)

Set the sampling parameters, time_step, time_lag, NPP as desired.

time_step: amount of time between measurements of the position and velocity of the simulated reed; default is one driving period
recommended: $2\,\pi / \omega$

time_lag: amount of time to allow to elapse before making the first measurement; allows for settling into chaos basin; allows for sampling at different phases (simply allow part of a driving period to elapse before beginning measurements)
recommended: 100 time_step

NPP: number of points in phase space to measure.
recommended: 1000-4000

Execute the Parameters section (press enter with the cursor in the section)

Execute the Solve section; wait for the "computation done" output

note: the Solve section outputs the calculated sample data in the files "xduf.txt" and "vduf.txt" to be used in the same way as data files from the experiment (see section IV)

Execute the Plot section to view the x-v sample that you have calculated
(could be omitted)

Appendix B: Box-Counting Dimension

To compute the box-counting dimension, we lay a square grid on top of phase space, count the number of squares, N(r), of the grid that contain points of the Poincaré section (figure 1), and then repeat many times for grids of different sizes, r.

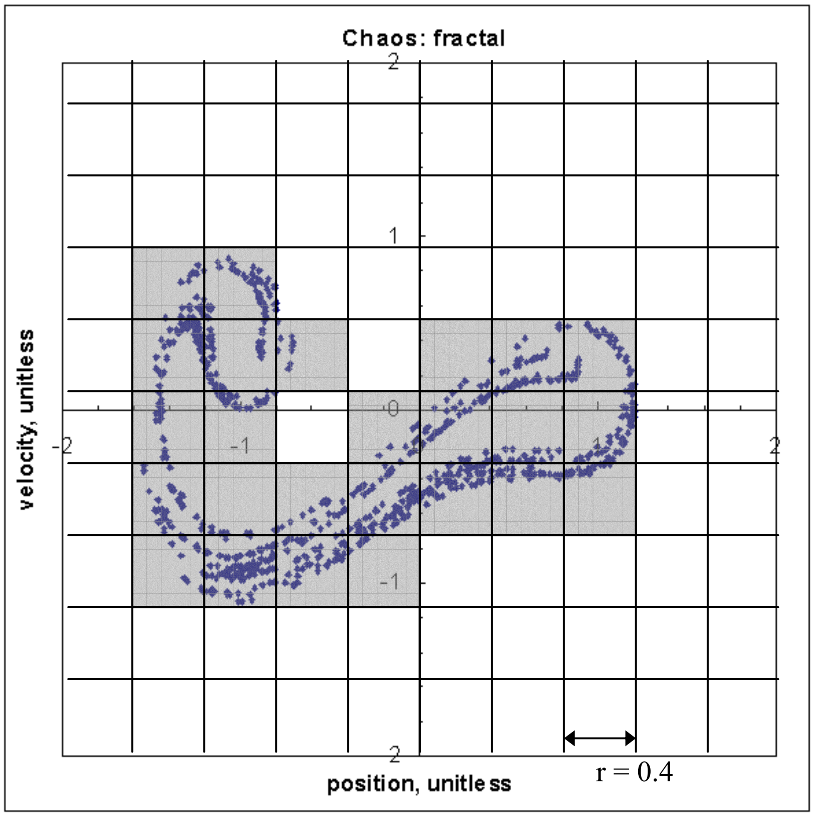

Figure 1. It takes 25 boxes of size 0.4 x 0.4 to cover this Poincaré section (i.e., N(0.4) = 25).

If the shape is 1-dimensional, a line, it can be shown that N(r) = 1/r. Similarly, if the shape is 2-dimensional, $N(r) = 1/r^2$, and if the shape is 3-dimensional, $N(r) = 1/r^3$. For a more complicated shape, the relation between N(r) and 1/r may not be so clear. It turns out (prove it to yourself if you'd really like proof) that

$$N(r) = k\,(1/r)^d \qquad (6)$$

where d is the dimension and k is an arbitrary constant. Taking the log of both sides, we get

$$\log(N(r)) = \log(k) + d\,\log(1/r) \qquad (7)$$

and so we can get d by plotting log(N(r)) vs. log(1/r) and fitting the slope.

Using MPL Box-Counting

1) Copy your x-v data into the obvious cells. Scale if desired. Note: it is probably easier to scale the data than it is to change box sizes and axes.

2) Check to see that all of your data is being plotted in the plot at the top right of the sheet

3) Set the box sizes (the r's) so that you get N(r)'s ranging from 10 to the number of data points. You will need to avoid using r's that are too small or too large.

4) Click the **boxcount** button to compute the N(r)'s that correspond to the r's that you have set.

5) Go to the **lsf** sheet, make sure that there are no extraneous data points in the data columns; the relevant data should have been written in by the **boxcount** macro.

6) Click the **fit** button to fit the data.

7) Use your judgment to determine whether or not each of the data points should be used in the fit and if additional data points should be included.

8) Note: errors cannot be reported. It is up to you to decide how good the fit is and how uncertain the parameters are. Now you know, … and knowing's half the battle!