Further Analysis of $\bar{p}p \to 3\pi^0$, $\eta\eta\pi^0$ and $\eta\pi^0\pi^0$ at Rest

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Abstract

A fresh analysis is reported of high statistics Crystal Barrel data on $\bar{p}p \to 3\pi^0$, $\eta\eta\pi^0$, $\eta\pi^0\pi^0$ and $\eta\eta'\pi^0$ at rest. This analysis is made fully consistent with CERN-Munich data on $\pi^+\pi^- \to \pi^+\pi^-$ up to a mass of 1900 MeV, with GAMS data on $\pi^+\pi^- \to \pi^0\pi^0$, and with BNL and ANL data on $\pi^+\pi^- \to K\bar{K}$, which are fitted simultaneously. There is evidence for an $I = 0$ $J^{PC} = 2^{++}$ resonance with weak ($\leq 7\%$) coupling to $\pi\pi$, strong coupling to both $\rho\rho$ and $\omega\omega$ and pole position $1534 - i90$ MeV. This resonance agrees qualitatively with GAMS and VES data on $\pi\pi \to \omega\omega$, previously interpreted in terms of a resonance at 1590 – 1640 MeV. New masses and widths for (A) $f_0(1370)$ and (B) $f_0(1525)$, fitted to all eight data sets, are $M_A = 1300 \pm 15$ MeV, $\Gamma_A = 230 \pm 15$ MeV, $M_B = 1500 \pm 8$ MeV, $\Gamma_B = 132 \pm 15$ MeV. Branching ratios to $\pi\pi$ and $\eta\eta$ are given, and differ significantly from earlier determinations because of a new procedure.

1 Introduction

In a series of publications [1-8], high statistics data from Crystal Barrel on $\bar{p}p \to 3\pi^0$, $\eta\eta\pi^0$, $\eta\pi^0\pi^0$ and $\eta\eta'\pi^0$ at rest have been analysed and shown to require the presence of $I=0$, $J^{PC}=0^{++}$ resonances at 1300-1370 MeV and 1500-1525 MeV. We shall refer to these resonances according to the masses quoted most recently by the Particle Data Group [9], namely $f_0(1370)$ and $f_0(1525)$. However, we now find significantly lower mass for the first one. The $f_0(1525)$ has also been observed in J/Ψ radiative decays in the 4π channel [10].

In all our previous publications, a problem has been that $f_0(1370)$ and $f_0(1525)$, which have been fitted to Crystal Barrel data, have been inconsistent with published analyses of CERN-Munich (CM) data [11]. This problem has now been resolved by a fresh analysis of the latter [12]. That analysis includes both $f_0(1370)$ and $f_0(1525)$ consistently in fitting all channels. It also fits simultaneously GAMS data on $\pi^+\pi^- \to \pi^0\pi^0$ [13], and BNL [14] and ANL [15] data on $\pi^-\pi^+ \to K\bar{K}$. This has prompted us to refine our fits to Crystal Barrel data on $\bar{p}p \to 3\pi^0$, $\eta\eta\pi^0$ and $\eta\pi^0\pi^0$ at rest.

We first scrutinise $f_0(1370)$ to check that its effects cannot be explained in any other way. It is needed to fit all three sets of Crystal Barrel data, particularly $3\pi^0$. This is so even if P-state annihilation to 2^+ final states is added freely to the analysis; this is one topic we shall present. We discuss $f_0(1525)$ very little, since it is already clearly established in many data sets;

we are able to refine its parameters slightly. We are also able to refine estimates of branching ratios to $\pi\pi$ and $\eta\eta$ channels. This is important, since $f_0(1525)$ is now viewed as a candidate for the lowest 0^+ glueball [16,17].

The new development reported here is that we are able to pin down more precisely the $I = 0$ $\pi\pi$ $J^{PC} = 2^{++}$ amplitude. We now find evidence that the $\omega\omega$ threshold plays a strong role in this amplitude. Including the $\omega\omega$ threshold, we are able to build up a consistent picture of a resonance at ~ 1540 MeV coupling to $\pi\pi$ (weakly), $\rho\rho$ and $\omega\omega$. The GAMS group has claimed a resonance, $f_2(1640)$ from $\omega\omega$ data [18] and their observations have been confirmed by VES [19] with a slightly lower mass, $M = 1590 \pm 30$ MeV, $\Gamma = 100 \pm 20$ MeV. These results may now be interpreted in terms of a single 2^+ resonance, whose high mass tail accounts for the GAMS and VES signals in $\omega\omega$.

The layout of the paper is as follows. Section 2 reviews the $\pi\pi$ S-wave and presents formulae briefly. Section 3 outlines the content of the Dalitz plots and Section 4 gives our new fits to the data. In particular, the evidence for $f_0(1370)$ is discussed in some detail. Section 5 discusses evidence for the 2^+ resonance at 1540 MeV. Section 6 discusses branching ratios and Section 7 gives our conclusions.

2 Survey of the $\pi\pi$ S-wave

We begin with a qualitative discussion of the $\pi\pi$ S-wave amplitude, since there appears to be widespread confusion about the number and nature of the poles in this amplitude.

Fig. 1(a) shows the intensity of the S-wave amplitude for $\pi^+\pi^- \to \pi^+\pi^-$, taken from the new analysis of ref. [12]. The corresponding phase shift is shown in Fig. 1(b). The $f_0(980)$ resonance makes a narrow dip in the intensity just below 1 GeV, because of destructive interference with the remaining S-wave component. At $\sim$ 1500 MeV, the $f_0(1525)$ produces a second deep minimum. It is important to realise that $f_0(980)$ and $f_0(1525)$ resonances appear as dips rather than peaks, because of interference with a slowly varying component. For brevity, we shall refer to this as a 'background' amplitude in order to distinguish it from narrow resonances.

This background component of the amplitude has been parametrised in all analyses by a pole far from the physical axis. Here we find a pole position of $1247 - i588$ MeV on the second sheet and $1221 - i563$ MeV on the third. Earlier, Au, Morgan and Pennington [20] found $910 - i350$ MeV, averaged over two similar solutions. Zou and Bugg [21] explain a large part of the slowly varying component in terms of t and u-channel exchange due to ρ and $f_2(1270)$ on the left-hand cut. The Particle Data group has for many years listed $f_0(1200 - 1400)$ as a highly elastic resonance. They have identified it with the broad bump in Fig. 1(a) between the $f_0(980)$ and $f_0(1525)$ dips.

This preamble is required in order to differentiate the $f_0(1370)$ resonance. It is *not* the same as the PDG $f_0(1200-1400)$. Neither is it the same as the distant singularity which we have just described in the mass range 940 – 1200 MeV. Instead, it is a rather inelastic resonance, for which we now find $M = 1300 \pm 15$ MeV, $\Gamma_{2\pi} = 60 \pm 20$ MeV, $\Gamma_{total} = 230 \pm 15$ MeV. It is responsible for the small dip at 1.3 GeV in Fig. 1(a).

The parametrisation of the $\pi\pi$ S-wave amplitude is discussed at length in ref. [12], to which we refer for detailed formulae. Here we use the form labelled B in that paper, using the K-matrix. It describes in a unified way both $f_0(980)$ and the slowly varying part of the S-wave amplitude. Suppose the K-matrix elements describing $\pi\pi \to \pi\pi$ and $\pi\pi \to K\bar{K}$ are respectively K_{11} and K_{12}. Then the annihilation amplitude to the $\pi\pi$ final state in $\bar{p}p \to 3\pi^0$ and $\eta\pi^0\pi^0$ is written:

$$f_{\pi\pi} = (\Lambda_1 + \Lambda_2 s)K_{11} + \Lambda_3 K_{12}. \tag{1}$$

This amplitude contains the poles of the $\pi\pi$ amplitude, but has a different numerator arising from coupling to $\bar{p}p$. The coupling constants Λ are complex. The theoretical background to equn. (1) is discussed in full in ref. [3]. The first term describes direct coupling from $\bar{p}p$ through an intermediate $\pi\pi$ state; we find it necessary to give the amplitude a linear s-dependence to achieve a good fit over the large mass range. The last term describes coupling from $\bar{p}p$ to $\pi\pi$ through $K\bar{K}$ intermediate states. Coupling to the $\eta\eta$ final state in $\bar{p}p \to \eta\eta\pi^0$ is described by equn. (1), but setting $\Lambda_2 = 0$

because of the narrow range of s covered by the $\eta\eta\pi$ data.

Resonances other than $f_0(980)$ are described by Breit-Wigner amplitudes:

$$f = \frac{\Lambda}{M^2 - s - iM\Gamma}, \tag{2}$$

where Λ is a complex coupling constant and $\Gamma =$ constant for most resonances. We shall present some alternative fits which include the s-dependence of the coupling to the $K\bar{K}$ and 4π channels:

$$\begin{aligned} \Gamma &= \Gamma_{2\pi} + \Gamma_{KK}(s) + \Gamma_{4\pi}(s), && (3) \\ \Gamma_{KK}(s) &= A\sqrt{1 - 4M_K^2/s} \quad \text{for} \quad s > 4M_K^2, && (4) \\ &= iA\sqrt{4M_K^2/s - 1} \quad \text{for} \quad s < 4M_K^2; && (5) \end{aligned}$$

we take $\Gamma_{4\pi}$ proportional to $\rho\rho$ phase space, integrated over the shape of each ρ. The dependence of $\Gamma_{2\pi}$ on 2π phase space is included, but in practice the s-dependence is negligible. For the $\pi\pi$ D-wave, we find it necessary to include a second resonance at ~ 1540 MeV, coupling strongly to the $\omega\omega$ channel:

$$\Gamma = \Gamma_{2\pi} + \Gamma_{\rho\rho}(s) + \Gamma_{\omega\omega}(s). \tag{6}$$

Explicit expressions parametrising $\Gamma_{\rho\rho}$ and $\Gamma_{\omega\omega}$ are given in Section 4.

3 Discussion of Dalitz plots

Dalitz plots for the three principal sets of annihilation data are shown in Fig. 2, namely for $3\pi^0$, $\eta\pi^0\pi^0$ and $\eta\eta\pi^0$. We shall interpret them in terms of 2-body processes $\bar{p}p \to \pi X$ or ηX, $X \to \pi\pi$ or $\eta\pi$ or $\eta\eta$. The first point to realise is that every channel has a large component coming from the slowly varying S-wave component in $\pi\pi$ or $\eta\eta$. (The $\eta\eta$ S-wave has the same denominator as $\pi\pi$). Typically it amounts to 50% of the cross section when integrated over the Dalitz plot. The foreground resonances stand out against the background, but interfere with it. There are then strong correlations between the form of the background and coupling constants of resonances. There is some lesser correlation between the form of the background and resonance masses and widths. Only narrow structures can be identified with confidence. The fit to the background needs to be flexible.

Because this background spans a very wide mass range from threshold to 1.74 GeV, one must anticipate that the coupling to $\bar{p}p$ may vary with s. We approximate this variation with a linear dependence. This is an assumption which is consistent with the data; but the true s-dependence could be more complicated. For $3\pi^0$ data and $\eta\pi^0\pi^0$, there are some conspicuous features which tie down the background amplitude well. For example, in Fig. 2(a) weak dips are visible across the whole extent of the Dalitz plot due to $f_0(980)$, labelled D. Also the strong, broad peak in the $\pi\pi$ S-wave at 800 MeV is responsible for most of the peak at $M \simeq 1560$ MeV labelled C; this peak arises from interferences between the tail of the nearby $f_0(1525)$, labelled B, and the S-wave in the second and third channels, which cross at this point. A further strong feature is a 'hole' at the centre of the Dalitz plot. In order to fit this, it is essential to have a large cancellation between the terms Λ_1 and $\Lambda_2 s$ of equn. (1) at this mass. These features determine the background rather closely, but there is still some flexibility in relative contributions from low and high masses, because of the interferences between s, t and u channels; for illustrations of the degree of flexibility, see Figs. 18 and 19 of ref. [3].

Next consider the $\eta\pi^0\pi^0$ channel. In Fig. 2(b), there are conspicuous signals due to $a_2(1320) \rightarrow \eta\pi$, $a_0(980) \rightarrow \eta\pi$ and $f_0(980) \rightarrow \pi\pi$. In this case, a small intensity is observed at the top left and bottom right corners of the Dalitz plot. This puts limits on the background amplitude for low $M_{\pi\pi}$; together with the strong $f_0(980)$ contribution (which this time appears as a peak, because of interference with the $\eta\pi$ S-wave), it again pins down the background amplitude well. It turns out that the lower left corner of this Dalitz plot can only be fitted well with inclusion of $f_0(1370) \rightarrow \pi\pi$.

However for $\eta\eta\pi$, Fig. 2(c), there are no conspicuous features from the background amplitude, but it is nonetheless present. Those features which are observed come from $a_0(980)$, $f_0(1370)$ and $f_0(1525)$. In fitting the $\eta\eta\pi$ data, there is consequently a strong correlation between resonances and background. There is the additional possibility that the upper tail of $f_0(980)$ may contribute an unknown amount to the $\eta\eta$ channel near threshold.

A further point is that triangle graphs may also be significant and may spoil the simple isobar model. They involve production and decay of a resonance:

$$\bar{p}p \to A + X, \qquad X \to B + C,$$

followed by rescattering of one of the decay products B or C from the spectator A. Such amplitudes vary logarithmically with s [3] and may be confused with the broad background in the $\pi\pi$ or $\eta\eta$ S-wave.

In summary, there are uncertainties in the background amplitude in each channel, and a combined analysis of several channels gives results for foreground resonances which are much more secure than separate analyses of individual channels. Some of the branching ratios and masses we quote here have changed significantly from the earlier publications for this reason.

4 Fits to Data

We begin our discussion by assuming that annihilation occurs only from the initial $\bar{p}p$ $^1S_0$ state. Then we shall add annihilation from initial $^3P_2$ and $^3P_1$ states. The data which are fitted simultaneously are as follows: Crystal Barrel data on $\bar{p}p \to 3\pi^0$, $\eta\pi^0\pi^0$, $\eta\eta\pi^0$ and $\eta\eta'\pi^0$ at rest; CERN-Munich data on $\pi^-\pi^+ \to \pi^-\pi^+$; GAMS data on $\pi^-\pi^+ \to \pi^0\pi^0$; and ANL and BNL data on the magnitude and phase of the S-wave amplitude for $\pi^-\pi^+ \to K\bar{K}$.

There is a technical comment concerning the last set of data. The ANL and BNL data on $\pi^-\pi^+ \to K^0_S K^0_S$ differ in normalisation by a factor 1.5. We have chosen to use the normalisation given by ANL for two reasons. Firstly, the ANL group has cross-checked the normalisation with further data on $\pi^-\pi^+ \to K^-K^+$. Secondly, the fit is marginally better with this normalisation. However, in first approximation the normalisation affects only the coupling strength of $f_0(1370)$ to $\pi\pi$ and $K\bar{K}$ channels, and has little effect on the mass and width fitted to this resonance.

With only S-state annihilation, our best fit gives the contributions to χ^2 shown in column (a) of Table 1. (In the actual fitting procedure, data sets are weighted so that each makes a similar contribution, to prevent high statistics channels overwhelming those with lower statistics; results are insensitive to the precise weighting). Argand diagrams for the $\pi\pi$, $\eta\eta$ and $\pi\eta$ S-waves in $\bar{p}p$ annihilation are shown in Fig. 3. For $3\pi^0$ data and $\eta\eta\pi^0$, Figs. 3(a) and (b), there are two distinct loops near 1300 and 1500 MeV, giving a minimum in the intensity at $\sim$ 1420 MeV.

4.1 Discussion of $f_0(1370)$ in $3\pi^0$

The $f_0(1370)$, which we shall fit here with a mass of 1300 MeV, is obscured in $3\pi^0$ data by the $f_2(1270)$ signal. In $\eta\eta\pi^0$ data it is weak. We need to examine whether it could be an artefact of the way the background amplitude is fitted; or whether it could be removed by including P-state annihilation. We have tried a variety of ways of removing one of the two loops of Fig. 3, or explaining it in a different way, in order to eliminate $f_0(1370)$. All these attempts have failed, and we now outline them.

Firstly, one notices that the intensity of the amplitude falls to a minimum at 1420 MeV. Could it be that the broad background amplitude, which we discussed qualitatively in Section 2, is being killed by the onset of inelasticity? That is, could the amplitude be tending to the origin of Fig. 3(a) before $f_0(1525)$ drives it to large values again? Attempts in this direction fail completely. We have dropped $f_0(1370)$ from the fit, and introduced rapidly increasing inelasticity into the background amplitude. The result is a large increase in χ^2 (> 400, compared to the 5 expected statistically). The reason is not hard to find. Fig. 4 shows the phase space for $\rho\rho$ and $\sigma\sigma$ final states, which are the most likely inelastic channels. (Here σ stands for the full $\pi\pi$ S-wave amplitude). Phase space is parametrised by equn. (40) of ref [12]. Also shown is a full curve which approximates the $\pi\pi \to 4\pi$ cross section measured by Alston-Garnjost et al. [22] in $\pi^+ p \to \Delta^{++}(4\pi)$. The available phase space rises extremely slowly from 10% at $\sim$ 1500 MeV to $\sim$ 90% at $\sim$ 2000 MeV. The inelasticity cannot be large enough or vary fast enough to account for the minimum at 1420 MeV in Figs. 3(a) or (b).

Next we have considered the possibility of fitting without $f_0(1370)$, but including $f_0(1525)$ and in addition $f_0(1750)$ [10]. The convergence of the fit is *very* much worse than with inclusion of $f_0(1370)$ and it is clear that a vital element is missing. Nonetheless, the Argand diagram for this fit to $3\pi^0$ data, shown in Fig. 5, has a certain resemblance to that of Fig. 3(a). The $f_0(1525)$ and $f_0(1750)$ resonances have conspired to reproduce the same double-loop structure as in Fig. 3, but quantitatively the fit to the data has deteriorated enormously (by $\Delta\chi^2 > 300$). One particular respect in which it has deteriorated is in the fit to GAMS data on $\pi^-\pi^+ \to \pi^0\pi^0$. Fig. 6 shows the fit to those data at small t, with and without $f_0(1370)$ in the fit; in (b), without this resonance, the fit is very bad.

Next we consider P-state annihilation. We find that the mass of $f_0(1370)$ in our best fit has moved rather close to that of $f_2(1270)$. Could it be that $f_2(1270)$ is being produced from initial $^3P_2$ and $^3P_1$ states, and that our fit is mocking up this effect by means of $f_0(1370)$? We have tried eliminating $f_0(1370)$ and including annihilation from initial P-states to either or both of $f_2(1270)$ and $f_2(1540)$. We shall discuss this P-state annihilation in detail below. However, the conclusion will be that, even with the maximum flexibility in the P-state contributions, the fit is worse than our best fit by ~ 300; the increase comes almost entirely from $3\pi^0$ data. The discrimination against this alternative arises essentially from the fact that the $3\pi^0$ data respond to interferences between all three s, t and u channels, and serve therefore as a delicate interferometer.

4.2 $f_0(1370)$ in other annihilation channels

If $f_0(1370)$ is removed from the fit to $\eta\eta\pi^0$ data, χ^2 increases by 187.6, a highly significant amount. However, the mass and width of the resonance are not well determined by those data, because of strong correlations with the background. If it is removed from the fit to $\eta\pi^0\pi^0$ data, χ^2 increases by 827, an even larger amount, but again the mass is not well determined. This is because the $\eta\pi$ mass range extends only to 1328 MeV. These data restrict the width on the lower side of the resonance to a value in the region 150-250 MeV, i.e. not too wide.

4.3 $f_0(1370)$ summary

The data which really determine the mass and width of $f_0(1370)$ accurately are $3\pi^0$ and, to a lesser extent, $\pi\pi \to K\bar{K}$. A fit to $3\pi^0$ data alone, including P-state annihilation to $f_2(1270)$, gives $M = 1297$ MeV, $\Gamma = 222$ MeV, with statistical errors of a few MeV. Variations of the form chosen for the background amplitude give rise to variations of up to ± 15 MeV in both the mass and width. If the P-state annihilation is dropped, the mass goes down by 5 MeV and the width hardly changes. With P-state annihilation to $f_2(1270)$ included, the fit is stable and χ^2 of $3\pi^0$ data improves by a modest 19 for two extra parameters. The amount of annihilation is 3.9% from $^3P_1$ and 1.3% from $^3P_2$, compared with 9.5% to $f_2(1270)$ from the S-state. Adding P-state annihilation to $f_2(1540)$ makes the fit unstable, and will be discussed in detail below.

A free fit to data on $\pi^-\pi^+ \to K\bar{K}$ alone gives $M = 1299 - 1302$ MeV, $\Gamma = 222 - 239$ MeV, depending on decay channels assumed for $f_0(1525)$, hence the detailed expression for the s-dependence of its width. There is a clear peak visible by eye in $\pi\pi \to K\bar{K}$ data at 1300 MeV; it is the best visual evidence for this resonance. A combined fit to all data gives $M = 1298$ MeV, $\Gamma = 230$ MeV, with $3\pi^0$ data affecting the overall χ^2 by a factor 2.5 as much as $\pi\pi \to K\bar{K}$. In conclusion, all three data sets for annihilation to $3\pi^0$, $\eta\eta\pi^0$

and $\eta\pi^0\pi^0$ require the presence of $f_0(1370)$ rather strongly, but with a mass we now assess as 1300 MeV.

The fit (a) of Table 1 uses a constant width for $f_0(1370)$. Next we consider explicit s dependence using equns. (3-5) of Section 2; here $\Gamma_{KK}(s)$ is taken proportional to $K\bar{K}$ phase space and $\Gamma_{4\pi}(s)$ is taken proportional to $\rho\rho$ phase space. The fit improves slightly, as shown in the second column of Table 1. The pole position is almost identical to that obtained with constant width. The s-dependence of $\Gamma_{4\pi}$ makes interpretation difficult unless one does a rather lengthy calculation. In Table 2 we show masses and widths from fit (a), where $\Gamma_{4\pi}$ is taken as a constant. Although this fit has a slightly worse χ^2, the appearance of amplitudes on the Argand diagram is very close to that of fit (b), and the parameters are easier to understand. A full discussion of the s dependence of $\Gamma_{4\pi}$ is given in ref. [12].

The χ^2 of the present fits is superior to those reported in the most recent analysis of Crystal Barrel data [8], which used the K-matrix approach. For example, for $3\pi^0$ data χ^2 has improved from 2448 to 2007 in (a) or 1976 in (b).

5 Evidence for a 2^+ Resonance at 1540 MeV

In the past, there have been several pieces of evidence for an $I = 0$ $J^P = 2^+$ resonance in this mass range. Initially, Gray et al. [23] observed the existence of a strong resonance at 1527 MeV, but were unable to distinguish clearly between $J^P = 0^+$ and 2^+. It is quite possible that they were observing $f_0(1525)$. Later, the Asterix group provided evidence for AX(1565) [24,25]. Our own early analysis of $\bar{p}p \to 3\pi^0$ required a 2^+ contribution which was parametrised as a resonance at 1566^{+80}_{-50} MeV with $\Gamma \simeq 165$ MeV [2]. However, the amplitude was small, and we could not be confident that this resonance was distinct from the low energy tail of $f_2(1640)$ observed by GAMS [18] and VES [19]. At the recent Hadron'95 conference, the VES group presented evidence [26] for a 2^+ $I = 0$ resonance at 1540 MeV in the $\rho\rho$ channel, with a width of 150 MeV.

5.1 $f_2(1540)$ in $3\pi^0$ data

In our present fit to $3\pi^0$ data, there is a definite need for a 2^+ resonance around 1540 MeV if we fit with only S-state annihilation. If it is omitted, χ^2 increases by > 1000, a huge amount. Although the $f_2(1540)$ contributes only 20% as much as $f_0(1525)$ to the integrated cross section, interferences with $f_2(1270)$ and $f_0(1525)$ play a crucial role in fitting the data, particularly a dip crossing the edge of Fig. 2(a) at $M \sim 1420$ MeV. This dip is responsible for minima in both the $\pi\pi$ S and D-waves near this mass. The Argand diagram for the fitted D-wave is shown in Fig. 7.

A new feature is that, with the increased statistics now available, a definite cusp is required in this amplitude at the $\omega\omega$ threshold. Including this threshold for the $\pi\pi$ D-wave in $3\pi^0$ data improves χ^2 by 60 with the addition of just one extra parameter, namely $\Gamma_{\omega\omega}$. This is a significant amount, nearly

8σ. Contributions to χ^2 are shown in column (c) of Table 1. Most of the improvement comes from $3\pi^0$ data.

For this resonance, we use equations (2) and (6) of Section 2. We have made fits with a variety of forms for $\Gamma_{\rho\rho}(s)$ and all give very similar results. With $\Gamma_{\rho\rho}(s)$ given by the dashed curve of Fig. 3(a), the pole position is $(1534 \pm 20) - i(90 \pm 30)$ MeV. A convenient empirical parametrisation is

$$\Gamma_{\rho\rho}(s) = C\frac{\sqrt{1 - 16m_\pi^2/s}}{1 + \exp(D(s_0 - s))}, \tag{7}$$

with C constant, $D = 2.8$ GeV^{-2} and $s_0 = 2.846$ GeV^2. The denominator is a Fermi function which approximates the rise of the cross section with s. The numerator provides a cut-off at the 4π threshold, but in practice has negligible effect on the fits. For $\Gamma_{\omega\omega}$ we use

$$\begin{aligned} \Gamma_{\omega\omega} &= B\sqrt{1 - 4M_\omega^2/s} \qquad \text{for} \qquad s > 4M_\omega^2, & (8)\\ &= iB\sqrt{4M_\omega^2/s - 1} \qquad \text{for} \qquad s < 4M_\omega^2. & (9) \end{aligned}$$

With this form, equns. (2), (6) and (7-9) fit the data with $M = 1571$ MeV, $\Gamma_{2\pi} = 30$ MeV, $C = 2032$ MeV and $B = 500$ MeV. Here $\Gamma_{2\pi}$ is really only an upper limit set by CERN-Munich data, and B and C are rather strongly correlated, so individual parameter values have sizeable errors. On resonance, the 4π width given by equn. (7) is 408 MeV. For a $\bar{q}q$ resonance, simple counting of $q\bar{q}$ charges predicts $\rho\rho$ to be three times as large as to $\omega\omega$, i.e. $C = 3B$. A free fit gives $C = 4B$, but these parameters are closely correlated, and χ^2 is almost as good with $C = 3B$.

The Argand diagram of Fig. 7 shows two important features: (a) a small loop at about 1420-1430 MeV, (b) a strong cusp at 1564 MeV, the $\omega\omega$ threshold. It is the loop at 1420 MeV which really demands the presence of a second resonance in addition to $f_2(1270)$. Without a second resonance, there is no structure at this mass. It is the cusp at 1564 MeV which requires that the resonance couples strongly to $\omega\omega$. The 4π and $\omega\omega$ widths for this resonance are large, so it is intrinsically a rather broad resonance; however, it appears narrow because of the cusp at the $\omega\omega$ threshold.

5.2 Other evidence for $f_2(1540)$

Fig. 8 shows the predicted coupling of all channels. In $\pi\pi \to \omega\omega$ there is a sharp rise at threshold and a peak at $\sim$ 1610 MeV. This agrees with the peak mass observed by GAMS [18] and VES [19] in the $\omega\omega$ channel. Their data show a narrow peak, with the amplitude falling slightly more rapidly than our prediction above 1640 MeV. However, it is easy to reproduce this fall by including a modest form factor $\exp(-3p_\omega^2)$ in $\Gamma_{\omega\omega}$ of equn. (8). Here, p_ω is the momentum of the ω in Gev/c. We suggest that the resonance we fit to $3\pi^0$ data is the same object as $f_2(1640)$; in the $\omega\omega$ channel, the lower side of the resonance is below threshold. We note that there is also tentative evidence in WA91 data [27] for a 4π peak in the channel $a_2(1320)\pi$ at about this mass.

5.3 Is $f_2(1540)$ distinct from $f_2(1270)$?

As an alternative to a second 2^+ resonance at 1540 MeV, we have tried introducing an increasing inelasticity into $f_2(1270)$ due to the $\rho\rho$ channel. This is done by taking

$$\Gamma^{1270} = \Gamma_{2\pi} + \Gamma_{KK} + \Gamma_{\rho\rho}(s). \tag{10}$$

This fails completely to fit the data. It results in an amplitude spiralling slowly inwards above 1300 MeV on Fig. 7, but failing to reproduce the dip in the intensity at 1420 MeV and also failing to fit the vicinity of the $\omega\omega$ threshold.

5.4 P-state annihilation

There is one ambiguity which cannot be resolved without recourse to other published data. It is possible that $f_2(1540)$ is produced from initial $^3P_1$ and $^3P_2$ states as well as $^1S_0$. When we try adding this possibility to the fit to $3\pi^0$ data, the solution becomes rather unstable. If the mass of the 2^+ state is left free, it drifts down to about 1480 MeV. The total P-state contribution becomes 8.4% from $^3P_1$ and 12.5% from $^3P_2$. However, the convergence of the fit is excessively slow and this is a warning of instability. The value of χ^2 does improve by 98 for the inclusion of four extra parameters. The fit to the background amplitude changes by a large amount.

The contribution of $f_2(1540)$ from S-state annihilation to one sextant of the Dalitz plot is shown on Fig. 9(a). It is of course strongly peaked at the edge by its decay angular dependence, $(3\cos^2\theta - 1)^2$. Fig. 9(b) shows the full P-state contribution to the Dalitz plot when P-state annihilation is included to both $f_2(1270)$ and $f_2(1540)$. Here there is a problem. The Asterix group [25] has presented high statistics data on $\bar{p}p \to \pi^+\pi^-\pi^0$ in hydrogen gas, with an X-ray trigger which identifies P-state annihilation. Fig. 9(b) differs seriously from the P-state annihilation they show in their Fig. 1. Their data show well defined bands at 1270 and 1500-1560 MeV. ¿From their Table 3, one finds an angular dependence for $f_0(1270)$ of $1+2.1\cos^2\theta$ (i.e. mostly $^3P_1$). Fig. 9(b) peaks much more strongly at the edge of the Dalitz plot. The strong peak near 1270 MeV in Fig. 9(b) arises from constructive interference with the low energy tail of $f_2(1540)$, but the signal near 1540 MeV itself is very weak. It therefore seems likely that a free fit including P-state annihilation to $f_2(1540)$ in $3\pi^0$ data is fixing up minor defects in the background amplitude and is not trustworthy.

To rectify this situation, we have made a simultaneous fit to the Asterix data taken with X-ray coincidence. The result is that P-state annihilation in $3\pi^0$ data is severely constrained. However, there is still a minor problem. Asterix data do not separate cleanly contributions between $f_2(1270)$ and $f_0(1300)$, nor between $f_0(1525)$ and $f_2(1540)$. As concerns fitting the intensity distribution of each band, this does not matter, since the effect is the same in $3\pi^0$ data to first order. But, as one sees in Fig. 2(a), the $f_2(1270)$ bands (labelled A) from different channels meet at the edge of the Dalitz plot and interfere there. In Asterix data on $\pi^+\pi^-\pi^0$, this interference is absent, since $f_2(1270)$ is present only in $\pi^+\pi^-$. So there is some ambiguity in transferring the information from $\pi^+\pi^-\pi^0$ data to $3\pi^0$. In practice, this gives rise only to a small range of uncertainty. However, in order to err on the side of safety, we shall finally make only semi-quantitative use of the Asterix data.

Asterix found a 23.1% contribution from $f_2(1270)$ and 8.3% from AX to P-state annihilation. We now find 19.2% contribution from $f_2(1270)$, 2.7% from $f_2(1540)$ and 5.6% from $f_0(1525)$. The small P-state annihilation to $f_2(1540)$ compared with $f_2(1270)$ is consistent with the large branching ratio of $f_2(1270)$ to $\pi\pi$, compared with that for $f_2(1540)$, which is $\leq 7\%$. In a simultaneous fit with other data, 250 Asterix points give a χ^2 of 435.9 if $f_2(1540)$ is omitted; adding $f_2(1540)$, χ^2 drops to 376.4 with the addition of four extra parameters. Using these amplitudes in fitting our $3\pi^0$ data, we find a total contribution of P-state annihilation in $3\pi^0$ data of $3.6 \pm 0.9\%$, mostly from ${}^3P_1 \to f_2(1270)$. This result is extremely close to a free fit to $3\pi^0$ data, using only P-state annihilation to $f_2(1270)$. In view of the minor problem discussed above, we regard the fit including P-state annihilation only to $f_2(1270)$ as the simplest and safest result at present, and it is what is shown in the Tables.

5.5 The 0^+ amplitude

We have checked the $\pi\pi$ S-wave for a cusp at the $\omega\omega$ threshold; the S-wave amplitude is a factor 5 larger than the 2^+ amplitude in $3\pi^0$ data. We find some tentative evidence for additional activity in the 0^+ amplitude in the mass range 1550-1700 MeV, but it is difficult to distinguish between alternative explanations. We have tried three things: (a) the effect of the $\omega\omega$ threshold, (b) G(1590) of GAMS, (c) $\theta(1710)$ with PDG values. All give small improvements to χ^2 in the range 13-40, and collectively an improvement of 80 for 5 extra parameters. The $\omega\omega$ threshold alone improves χ^2 by 13. Its effect on the Argand diagram of Fig. 3(a) is extremely small: barely more than the size of the points. For the present, we do not feel able to distinguish these three delicate effects, so we omit them all.

We do, however, have one remark. The proximity of the mass of $f_2(1540)$ to the $\omega\omega$ threshold suggests some connection with it. The box diagram for the process $\pi\pi \to \omega\omega \to \pi\pi$ with ρ exchange at each vertex gives rise to an attraction to the threshold. Other possible examples of such an effect are $f_0(975)$, $a_0(980)$, $f_1(1420)$, $\Lambda(1405)$ and the deuteron. For the $\pi\pi$ S-wave, the corresponding box diagram is a factor 6 larger at the $\omega\omega$ threshold than for 2^+. So the striking difference in behaviour between 0^+ and 2^+ channels at the $\omega\omega$ threshold is rather remarkable, and indicates that 0^+ couples weakly to $\omega\omega$.

6 Branching ratios

We now present a new way of assessing branching ratios of resonances that we believe is superior to previous methods. Let us take the channel $\bar{p}p \to 3\pi^0$ as an example. Here, a resonance like $f_0(1525)$ can be formed in three different ways, between particles 12, 23 and 13. In the past, branching ratios have been evaluated by fitting the data, then eliminating all contributions except those from this resonance, and forming the amplitude squared:

$$\frac{d\sigma}{d\Omega} = |f(12) + f(13) + f(23)|^2. \tag{11}$$

This procedure includes interferences between all three channels.

In $3\pi^0$ data, the resonance is formed in proximity to a spectator pion, and the interferences between the three channels play a strong role in actually fitting the data. However, what we need to know are the properties of an isolated resonance, decaying without interference with the spectator pion: the branching ratio of a resonance should be independent of its environment. This implies dropping the interferences between the three channels in evaluating branching ratios, i.e. using $3|f(12)|^2$. It makes quite a difference. For example, for $f_0(1525)$, it reduces the branching ratio to the $\pi\pi$ channel by $\sim 30\%$. This is because of the low energy tail of the resonance, which has a much larger phase space below resonance. Values of branching ratios evaluated in this way are given in Table 3. This table also summarises percentage contributions of each channel to $3\pi^0$, $\eta\pi^0\pi^0$ and $\eta\eta\pi^0$ data. In this paper, we do not discuss the fit to $\eta\pi^0\pi^0$ in detail, since it has changed little from previous work. However, for completeness, we list the latest branching ratios for this channel in Table 3.

In ref. [8], the enhancement at the $\eta\eta'$ threshold was described by a

Breit-Wigner amplitude with a full width Γ in the denominator proportional to q, the momentum of the η and η' in the rest frame of the resonance; see equn. (2) of ref. [8]. If we interpret this enhancement as the $\eta\eta'$ decay of the $f_0(1525)$, it is more realistic to take Γ to be constant, because of the additional open channels $\pi\pi$ and $\eta\eta$. In ref. [7], the fitted mass of the resonance was 1545 ± 25 MeV. This high mass reflected the fact that Γ went to zero at the $\eta\eta'$ threshold. Now, with Γ constant, the fitted mass comes out in the range $1480 - 1520$ MeV, according to the way in which experimental background is parametrised. It is entirely consistent with the mass of 1500 MeV fitted to other channels. There is no change in the branching ratio to the $\eta\eta'$ channel. The reason for this is that the data are fitted purely to $f_0(1525)$ plus an incoherent phase space background, and the latter does not change significantly.

The branching ratios given in Table 3 refer only to $\pi^0\pi^0$ final states. In evaluating $\Gamma_{2\pi}$, one needs to multiply by 3 to account for charged states in addition. In the work of ref. [12], $\Gamma_{2\pi}$ is determined from fits to CERN-Münich data and $\pi^-\pi^+ \to K\bar{K}$. Results are $\Gamma_{2\pi} = 60 \pm 12$ MeV for $f_0(1525)$ and $\Gamma_{2\pi} = 60 \pm 20$ MeV for $f_0(1370)$. Using these values and the branching ratios of Table 3, we deduce $\Gamma_{\eta\eta} = 4.7 \pm 0.8$ MeV, $\Gamma_{\eta\eta'} = 3.9 \pm 1.0$ MeV for $f_0(1525)$ and 1.1 ± 0.9 MeV for $f_0(1370)$. We will comment more fully below on possible unreliability in the result for $f_0(1370)$.

It is also of interest to factor out of $\Gamma_{\eta\eta}$ and $\Gamma_{2\pi}$ the phase space term

$\rho = 2q/\sqrt{s} = \sqrt{1-4m^2/s}$, where m is the mass of η or π. The result is proportional to the square of the coupling constant to each channel. Results are as follows:

$$\begin{array}{ccc} \frac{\Gamma_{2\pi}^{1500}}{\rho_{2\pi}} & : & \frac{\Gamma_{\eta\eta}^{1500}}{\rho_{\eta\eta}} : \frac{\Gamma_{\eta\eta'}^{1500}}{\rho_{\eta\eta'}} : \frac{\Gamma_{2\pi}^{1300}}{\rho_{2\pi}} : \frac{\Gamma_{\eta\eta}^{1300}}{\rho_{\eta\eta}} = \\ 3 & : & 0.34 \pm 0.06 : 1.55 \pm 0.39 : 3.02 \pm 1.17 : 0.10 \pm 0.08. \end{array} \tag{12}$$

The phase space factor for the $\eta\eta'$ channel has been integrated over the $f_0(1525)$ resonance.

The result for the $\eta\eta$ decay of $f_0(1525)$ is a factor 2 lower than was given in an earlier publication [8], partly because of the new way of doing the arithmetic, and partly due to a large change in the way the background has been fitted. This has led us to study the sensitivity of the branching ratios in the present work to the form of the background. Table 4 shows branching ratios from four fits where drastic changes have been made in the background. The first column shows our favoured fit with contributions to the background from K_{11} and K_{12} of equn. (1). In the second column, the latter term is dropped; χ^2 increases by a very large amount. In the third column, a simple background of the form $\Lambda_1 + \Lambda_2 s$ is used instead, i.e. ignoring information about the $\pi\pi$ amplitude as a function of s. This gives a rather worse fit than our favoured solution, but is not terrible. The fourth column shows results using a purely constant background.

The essential conclusion is that the branching ratios of the dominant signals $f_0(1525)$ and $a_0(980)$ are quite stable to very large changes in the background. However, the branching ratio to $f_0(1370)$ is not, and must be regarded as subject to a possible systematic error of a factor 2. Our opinion is that columns (a) and (c) of Table 4 span the range of likely values. In view of this uncertainty, one might think that $f_0(1370)$ may make no contribution to $\eta\eta\pi^0$ data at all. That is not the case. The data firmly demand a cusp or small loop in the Argand diagram of Fig. 3(b) at $\sim$ 1420 MeV. Without any $f_0(1370)$ contribution, it is not possible to reproduce this effect from the background alone; however, there is considerable flexibility in the way background and $f_0(1370)$ combine in both magnitude and phase.

We discussed in the previous section the contribution which P-state annihilation might make to $3\pi^0$ data. These contributions of course affect the branching ratios of $f_0(1370)$ and $f_0(1525)$ and Table 5 collects some numbers. In the first column, they are shown including P-state annihilation to $f_2(1270)$; these coincide with the values give in Table 3. In the second column of the Table, the product branching ratio is shown assuming pure S-state annihilation. The third column shows branching ratios from the fit including P-state annihilation to both $f_2(1270)$ and $f_2(1540)$; we do not regard this fit as satisfactory, but it indicates the maximum variation in branching ratios we have seen and a probable lower limit on the contribution from $f_0(1370)$. There is again some uncertainty in the branching ratio to $f_0(1370)$ in $3\pi^0$ data. We believe that the first column gives the most reliable result; it includes a small contribution from P-state annihilation to $f_2(1270)$, and the second column is likely to be an upper limit.

7 Conclusions

We have presented a fit to Crystal Barrel data on $\bar{p}p \to 3\pi^0$, $\eta\eta\pi^0$ and $\eta\pi^0\pi^0$ data that is now fully consistent with several other sets of data: CERN-Munich data on $\pi^-\pi^+ \to \pi^-\pi^+$, GAMS data on $\pi^-p \to \pi^0\pi^0 n$, and data from ANL and BNL on $\pi^-\pi^+ \to K\bar{K}$. This consistency is an important step forward. The amount of P-state annihilation is constrained by Asterix data on $\bar{p}p \to \pi^+\pi^-\pi^0$ from initial P-states. We present new determinations of masses, widths and branching ratios for $f_0(1525)$ and $f_0(1370)$. The latter resonance is definitely required by the data on $3\pi^0$ and $\eta\pi^0\pi^0$ final states, as well as $\pi\pi \to K\bar{K}$, but there is some uncertainty over its branching fractions, particularly in $\eta\eta\pi^0$ data.

In $3\pi^0$ data, there is now evidence for a cusp at the $\omega\omega$ threshold in the amplitude with $J^P = 2^+$. Our fitted amplitude has a pole at 1534 - i90 MeV. We suggest that this resonance is to be identified with the $f_2(1640)$ resonance of GAMS and VES. A straightforward explanation of $f_2(1540)$ is that it is the $q\bar{q}$ radial excitation of $f_2(1270)$.

8 Acknowledgements

We thank the technical staff of the LEAR machine group and of all the participating institutions for their invaluable contributions to the experiment.

We acknowledge financial support from the British Particle Physics and Astronomy Research Council, the German Bundesministerium für Bildung, Wissenschaft, Forschung und Technologie, the U.S. Department of Energy and the National Science Research Fund Committee of Hungary (contract No. DE-FG03-87ER40323, DE-AC03-76SF00098, DE-FG02-87ER40315 and OTKA F014357). N. Djaoshvili, F.-H. Heinsius and K.M. Crowe acknowledge support from the A. von Humboldt Foundation.

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Figure Captions

Fig. 1. (a) The intensity of the $\pi^+\pi^- \to \pi^+\pi^-$ S-wave amplitude against $\pi\pi$ mass, taken from ref. [12]; (b) the phase shift for this amplitude.

Fig. 2. Dalitz plots for (a) $\bar{p}p \to 3\pi^0$ at rest, from ref. [8], (b) $\eta\pi^0\pi^0$ [8], and (c) $\eta\eta\pi^0$ [7]. In (a), $M^2(\pi^0\pi^0)$ is plotted in units of MeV^2.

Fig. 3. Argand diagrams for S-wave contributions to our fit with only S-state annihilation: (a) the $\pi\pi$ S-wave in $3\pi^0$ data, (b) the $\eta\eta$ S-wave in $\eta\eta\pi^0$ data, (c) the $\pi\pi$ S-wave in $\eta\pi^0\pi^0$ data, and (d) the $\eta\pi$ S-wave in $\eta\pi^0\pi^0$ data. Masses are marked in Gev.

Fig. 4. Dependence of 4π phase space on M: (i) for $\rho\rho$ final states (dashed curve), (ii) for decays to $\sigma\sigma$ (dash-dot), and (iii) reproducing the cross section measured by Alston-Garnjost et al. [22].

Fig. 5. As Fig. 3(a) , but with $f_0(1370)$ removed from the fit, and replaced by $f_0(1750)$.

Fig. 6. The fit to GAMS data, ref. [13] on $\pi^- p \to \pi^+\pi^- n$ for $|t| < 0.2$ GeV^2 (a) including $f_0(1370)$ in the fit, (b) replacing it with $f_0(1750)$.

Fig. 7. The Argand diagram for the $\pi\pi$ D-wave fitted to $\bar{p}p \to 3\pi^0$ data.

Fig. 8. Predictions for $\pi\pi$, 4π and $\omega\omega$ coupled channels for $f_2(1540)$.

Fig. 9. (a) The contribution to one sextant of the Dalitz plot of $3\pi^0$ data from S-state annihilation to $f_2(1270)$; (b) the corresponding contribution from P-state annihilation to $f_2(1270)$ and $f_2(1540)$; (c) $3\pi^0$ data in liquid hydrogen. For visibility, (a) and (b) are scaled up in normalisation.

Data	No of points	(a) Γ constant	(b) $\Gamma_{4\pi}(s)$	(c) No $\omega\omega$
CM	705	1266	1260	1264
$3\pi^0$	1338	2007	1976	2057
$\eta\pi\pi$	3726	5131	5112	5132
$\eta\eta\pi$	1806	2550	2558	2558
GAMS($\|t\| < 0.2$)	126	184	184	187
GAMS($\|t\| > 0.3$)	105	148	148	148
$\pi\pi \to K^0_S K^0_S$	53	50	50	50
$\sigma(\pi\pi \to 4\pi)$	48	134	134	134
Total		11470	11422	11530

Table 1: Contributions to χ^2 from several fits to the data with S-state annihilation only: (a) best fit using constant widths for Breit-Wigner amplitudes, (b) with $\Gamma_{4\pi}$ proportional to $\rho\rho$ phase space, (c) without the $\omega\omega$ contribution to the 2^+ amplitude.

	M	$\Gamma_{\pi\pi}$	$\Gamma_{\eta\eta}$	Γ_{tot}
$f_0(1370)$	1300	60	1	230
$f_0(1525)$	1500	60	5	132
$f_2(1270)$	1276	177	1	195
$f_2(1540)$	1534	≤ 30	0	180

Table 2: Resonance parameters for fit (a) of Table 1.

	$\times 10^{-3}$	%
$\bar{p}p \to \pi^0 f_0(1370);\ f_0 \to \pi^0\pi^0$	0.64 $\pm$0.24	10.4
$\bar{p}p \to \pi^0 f_0(1370);\ f_0 \to \eta\eta$	0.036 $\pm$0.022	0.18
$\bar{p}p \to \pi^0 f_0(1525);\ f_0 \to \pi^0\pi^0$	0.82 $\pm$0.09	13.2
$\bar{p}p \to \pi^0 f_0(1525);\ f_0 \to \eta\eta$	0.191 $\pm$0.024	9.6
$\bar{p}p \to \pi^0 f_0(1525);\ f_0 \to \eta\eta'$	0.161 $\pm$0.040	100
$\bar{p}p \to \pi^0 f_2(1270);\ f_2 \to \pi^0\pi^0$	0.59 $\pm$0.06	9.5
$\bar{p}p \to \pi^0 f_2(1540);\ f_2 \to \pi^0\pi^0$	0.16 $\pm$0.06	2.5
$\bar{p}p \to \eta f_0(1300);\ f_0 \to \pi^0\pi^0$	0.12 $\pm$0.12	1.8
$\bar{p}p \to \eta f_2(1270);\ f_2 \to \pi^0\pi^0$	0.0020 $\pm$0.0008	0.03
$\bar{p}p \to \pi^0 a_0(980);\ a_0 \to \eta\pi^0$	0.69 $\pm$0.05	10.2
$\bar{p}p \to \eta a_0(980);\ a_0 \to \eta\pi^0$	0.284 $\pm$0.028	4.2
$\bar{p}p \to \pi^0 a_2(1320);\ a_2 \to \eta\pi^0$	1.97 $\pm$0.09	29.0
$\bar{p}p \to \eta a_2(1320);\ a_2 \to \eta\pi^0$	0.0017 $\pm$0.0005	0.03
$\bar{p}p \to \pi^0 a_0(1450);\ a_0 \to \eta\pi^0$	0.203 $\pm$0.038	3.0
$\bar{p}p \to \pi^0 a_2(1630);\ a_2 \to \eta\pi^0$	0.047 $\pm$0.006	0.07

Table 3: Product branching ratios for the best fit including P-state annihilation to $f_2(1270)$. Errors are assessed from systematic variations over a variety of fits. There are additional systematic errors of $\sim 10\%$ for branching ratios of each of the channels $3\pi^0$, $\eta\eta\pi 0$ and $\eta\pi^0\pi^0$. The last column shows the percentage contribution of each process to the data in which it appears.

			$\times 10^{-5}$	
	(a)	(b)	(c)	(d)
Background	11.6	4.8	9.7	5.4
$f_0(1370)$	3.6	22.7	7.3	15.1
$f_0(1525)$	19.1	14.1	20.4	14.9
$a_0(980)$	28.4	33.0	28.1	28.3
$a_2(1320)$	0.2	3.0	0.3	0.3
χ^2	2550	4156	2716	3286

Table 4: Product branching ratios in $\eta\eta\pi^0$ data for backgrounds fitted with (a) $\Lambda_1 K_{11}(s) + \Lambda_2 K_{12}(s)$, (b) $\Lambda_1 K_{11}(s)$, (c) $\Lambda_1 + \Lambda_2 s$, (d) constant.

	$\times 10^{-3}$		
	(a)	(b)	(c)
Background	4.06	3.96	3.24
$f_0(1370)$	0.645	0.884	0.327
$f_0(1525)$	0.818	0.851	0.909
$f_2(1270)$	0.586	0.524	0.559
$f_2(1540)$	0.158	0.161	0.130
${}^3P_1$	0.242	0	0.523
${}^3P_2$	0.079	0	0.773

Table 5: Product branching ratios for fits to $3\pi^0$ data (a) including P-state annihilation to $f_2(1270)$, (b) without P-state annihilation, (c) freely fitting P-state annihilation to both $f_2(1270)$ and $f_2(1540)$.