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A Primer on K-matrix Formalism
—Version VII—

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Abstract

A short description is given of the K-matrix formalism. The formalism, which is normally applied to two-body scattering processes, is generalized to production of two-body channels with final-state interactions.

A multi-channel treatment of production of $J^{PC} = 0^{++}$ and $J^{PC} = 2^{++}$ resonances has been worked out in the P-vector approach of Aitchison. An alternative approach, derived from the P-vector, gives the production amplitude as a product of the T-matrix for a two-body system and a vector Q specifying its production. This formulation, dubbed the Q-vector approach in this note, has also been worked out for several examples of practical importance.

Two separate 'derivations' of the Flatté formula are given in the K-matrix formalism, and a possible generalization of his formula is also pointed out. In addition, a derivation of David Bugg's formula for two resonances into two channels is given.

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1 Introduction

The K-matrix formalism provides an elegant way of expressing the unitarity of the S-matrix for the processes of the type $ab \to cd$. A concise description is given here for ease of reference, and its generalizations to arbitrary production processes are covered in some detail.

The reader is referred to the text book by Martin and Spearman[1] for some of the material covered in this note. However, one must note that the definitions given in this note are different from those used by Martin and Spearman. Cahn and Landshoff[2] and Au, Morgan and Pennington[3] have used the same definitions as those adopted in this note.

The unitary relationship involves a bilinear product, and one must exercise care with constant factors, as there is essentially no freedom with the coefficients. The derivation for the cross section from unitarity follows a well-defined prescription and, once defined, one must again adhere to it rigorously. The reader may note that a scrupulous attention has been given to these in this note.

2 S-Matrix and Unitarity

Consider a two-body scattering of the type $ab \to cd$. The differential cross section is given in terms of the invariant amplitude $\mathcal{M}$ and the 'scattering amplitude' f through

$$\frac{d\sigma_{fi}}{d\Omega} = \frac{1}{(8\pi)^2\, s}\left(\frac{q_f}{q_i}\right)|\mathcal{M}_{fi}|^2 = |f_{fi}(\Omega)|^2 \tag{1}$$

where 'i' and 'f' stand for the initial and final states; $\Omega = (\theta, \phi)$ denotes the usual spherical coordinate system; and $s = m^2$ is the square of the CM energy. The $q_i(q_f)$ is the breakup momentum in the initial(final) system. [The observed cross section is in reality the average of the initial spin states and the sum over all final spin states— this is suppressed here for simplicity.] The scattering amplitude can be expanded in terms of the partial-wave amplitudes

$$f_{fi}(\Omega) = \frac{1}{q_i}\sum_J (2J+1) T^J_{fi}(s) D^{J\,*}_{\lambda\mu}(\phi, \theta, 0) \tag{2}$$

where $\lambda = \lambda_a - \lambda_b$ and $\mu = \lambda_c - \lambda_d$ in terms of the helicities of the particles involved in the scattering $ab \to cd$. Note that this 'scattering amplitude' is a factor of two bigger than that with a more common definition (for example, see Section 5.1, Chung[4]). One may in addition note that the argument of the D-function is frequently given as $(\phi, \theta, -\phi)$ (see Jacob and Wick[5] and Martin and Spearman[1]). Integrating the differential cross section over the angles, one finds, for the cross section in the partial wave J,

$$\sigma^J_{fi} = \left(\frac{4\pi}{q_i^2}\right)(2J+1)|T^J_{fi}(s)|^2 \tag{3}$$

Note that T^J has no unit; the unit for the cross section is being carried by q_i^2. It is necessary to define more precisely the initial and the final states

$$\begin{aligned} |i\rangle &= |ab, JM\lambda_a\lambda_b\rangle \\ |f\rangle &= |cd, JM\lambda_c\lambda_d\rangle \end{aligned} \tag{4}$$

where M is the z-component of total spin J in a coordinate system fixed in the overall CM frame and the notations $\{ab\}$ and $\{cd\}$ designate additional informations needed to fully specify the initial and the final states. Because of conservation of angular momentum, an initial state in $|JM\rangle$ remains the same in the scattering process. Note the normalization (see Section 4.2, Chung[4])

$$\langle f|i\rangle = \delta_{ij} \tag{5}$$

In the remainder of this section and in subsequent sections, it is be understood that the ket states mentioned always refer to those of (4). In particular, explicit references to the total angular momentum J will be suppressed. Note that, with this convention, one has eliminated the necessity of specifying continuum variables such as angles and momenta.

In general, the amplitude that an initial state $|i\rangle$ will be found in the final state $|f\rangle$ is

$$S_{fi} = \langle f|S|i\rangle \tag{6}$$

where S is called the scattering operator. One may remove the probability that the initial and final states do not interact at all, by defining the transition operator T through

$$S = I + 2i\,T \tag{7}$$

where I is the identity operator. The factors 2 and i have been introduced for convenience. [One sometimes defines T via $S = I + iT$, without the factor 2 as was done by Jacob and Wick[5] or Martin and Spearman[1]— this is also responsible for the difference seen in (2).] From conservation of probability, one deduces that the scattering operator S is unitary, i.e.

$$S\,S^\dagger = S^\dagger S = I \tag{8}$$

From the unitarity of the S, one gets

$$T - T^\dagger = 2i\,T^\dagger T = 2i\,TT^\dagger \tag{9}$$

Or, in terms of the inverse operators, these can be rewritten

$$(T^\dagger)^{-1} - T^{-1} = 2iI \tag{10}$$

One may further transform this expression into

$$(T^{-1} + iI)^\dagger = T^{-1} + iI \tag{11}$$

One is now ready to introduce the K operator via

$$K^{-1} = T^{-1} + iI \tag{12}$$

From (11) one finds that the K operator is Hermitian, i.e.

$$K = K^\dagger \tag{13}$$

It can be shown from time-reversal invariance that the K operator is real, i.e. the K-matrix may be chosen to be real and symmetric. However, it is shown in the next section that this matrix becomes complex when it is analytically continued below a given threshold—the K-matrix nevertheless remains Hermitian, thus preserving unitarity of the S.

One can eliminate the inverse operators in (12) by multiplying by K and T from left and right and vice versa, to obtain

$$T = K + iTK = K + iKT \tag{14}$$

This shows that K and T operators commute, i.e.

$$[K, T] = 0 \tag{15}$$

and that, solving for T, one gets

$$T = K(I - iK)^{-1} = (I - iK)^{-1}K \tag{16}$$

Note that the T-matrix is complex only through the i that appears in this formula, i.e. T^{-1} has been explicitly broken up into its real and imaginary parts [see(12)].

Consider now an isoscalar $\pi\pi$ scattering in S-wave below $\sqrt{s} = 1\text{GeV}$. This is a single-channel problem and unitarity is rigorously maintained. From (8), one may set

$$S = e^{2i\delta} \tag{17}$$

where δ is the familiar phase shift. The transition amplitude T is given, from (7),

$$T = e^{i\delta} \sin\delta \tag{18}$$

Note that the factors 2 and i in (7) make the T attain the simple, familiar form. This formula shows that the trajectory of T in the complex plane (Argand diagram) is a circle of a unit diameter with its center at $(0, i/2)$. This is the so-called unitarity circle and the physically allowed T should remain at or within this circle. The S-wave cross section is, from (3),

$$\sigma = \left(\frac{4\pi}{q_i^2}\right) \sin^2\delta \tag{19}$$

The K-matrix for this case is simply

$$K = \tan\delta \tag{20}$$

A pole in K is therefore associated with $\delta = \pi/2$.

Consider next a two-channel problem in which both the K and T may be expressed as 2×2 matrices. Let

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \tag{21}$$

where $K_{12} = K_{21}$ and $K_{ij} =$ real. Then, from (12) one finds

$$T = \frac{1}{1 - D - i(K_{11} + K_{22})} \begin{pmatrix} K_{11} - iD & K_{12} \\ K_{21} & K_{22} - iD \end{pmatrix} \tag{22}$$

where

$$D = K_{11}K_{22} - K_{12}^2 \tag{23}$$

It is sometimes more convenient to parameterize the inverse of the K-matrix directly, as follows:

$$K^{-1} = U = \begin{pmatrix} U_{22} & -U_{12} \\ -U_{21} & U_{11} \end{pmatrix} \tag{24}$$

where $U_{12} = U_{21}$ and $U_{ij} =$ real. The T-matrix is now simpler,

$$T = \frac{\begin{pmatrix} U_{11} - i & U_{12} \\ U_{21} & U_{22} - i \end{pmatrix}}{U_{11}U_{22} - U_{12}^2 - 1 - i(U_{11} + U_{22})} \tag{25}$$

3 Lorentz-Invariant T-Matrix

The transition amplitudes T as defined in (7) is not Lorentz invariant. The invariant amplitude is defined through two-body wave functions for the initial and the final state, and the process of the derivation involves proper normalizations for the two-particle states (see Section 5.1, Chung[4]). The resulting invariant amplitude contains the inverse square-root of the two-body phase space elements in the initial and the final states. The Lorentz-invariant amplitude, denoted $\widehat{T}$, is thus given by

$$T_{ij} = \{\rho_i^*\}^{\frac{1}{2}} \ \widehat{T}_{ij} \ \{\rho_j\}^{\frac{1}{2}} \tag{26}$$

The indices i and j stand for the final and initial states; the complex conjugation for ρ_i is thus required for analytic continuation into the region below the i-th threshold. In matrix notation, one may write

$$T = \{\rho^\dagger\}^{\frac{1}{2}} \ \widehat{T} \ \{\rho\}^{\frac{1}{2}} \tag{27}$$

and

$$S = I + 2i \ \{\rho^\dagger\}^{\frac{1}{2}} \ \widehat{T} \ \{\rho\}^{\frac{1}{2}} \tag{28}$$

where the phase-space 'matrix' is diagonal by definition, i.e.

$$\rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \tag{29}$$

and

$$\rho_1 = \frac{2q_1}{m} \quad \text{and} \quad \rho_2 = \frac{2q_2}{m} \tag{30}$$

The q_i is the breakup momentum in channel i. (Here one considers a two-channel problem for simplicity.)

The cross section in the Jth partial wave is given by, from (3),

$$\sigma^J_{fi} = \left(\frac{16\pi}{s}\right)\left(\frac{\rho_f}{\rho_i}\right)(2J+1)|\widehat{T}^J_{fi}(s)|^2 \tag{31}$$

Note that this formula embodies the familiar presence of the flux factor of the initial system and the phase-space factor of the final system in the process $ab \to cd$. In the K-matrix formalism, one allows for ρ to become imaginary below a given threshold; however, the cross section above has no meaning below a threshold, and one could then modify the expression above by mutliplying it with two step functions: $\theta(\rho_i^2)$ and $\theta(\rho_f^2)$.

One may recapitulate the expressions for the differential cross section and its partial-wave expansion in terms of the invariant amplitudes $\widehat{T}^J_{fi}(s)$. For the purpose, one defines the 'invariant scattering amplitude'

$$\widehat{T}_{fi}(\Omega) = \sum_J (2J+1)\widehat{T}^J_{fi}(s) D^{J*}_{\lambda\mu}(\phi,\theta,0) \tag{32}$$

and the differential cross section is given by

$$\frac{d\sigma_{fi}}{d\Omega} = \left(\frac{4}{s}\right)\left(\frac{\rho_f}{\rho_i}\right)|\widehat{T}_{fi}(\Omega)|^2 \tag{33}$$

The initial and final density of states are, with $s = m^2$,

$$\begin{aligned} \rho_i &= \sqrt{\left[1-\left(\frac{m_a+m_b}{m}\right)^2\right]\left[1-\left(\frac{m_a-m_b}{m}\right)^2\right]} \\ \rho_f &= \sqrt{\left[1-\left(\frac{m_c+m_d}{m}\right)^2\right]\left[1-\left(\frac{m_c-m_d}{m}\right)^2\right]} \end{aligned} \tag{34}$$

in terms of the particle masses involved in the scattering $ab \to cd$. Note that these phase-space factors are normalized such that

$$\rho_i \to 1 \quad \text{as} \quad m^2 \to \infty \tag{35}$$

The invariant amplitude $\widehat{T}_{fi}(\Omega)$ is unitless, and has a partial-wave expansion (32). The partial-wave amplitude $\widehat{T}^J_{fi}(s)$ is related to the $\widehat{K}$-matrix via (39), and unitarity is preserved if the $\widehat{K}$-matrix is taken to be real and symmetric. It should be noted that the formula for the differential cross section (33) has no 'arbitrary' numerical factors. The 'conventional' invariant amplitude, introduced in (1), is given by

$$\mathcal{M}_{fi} = 16\pi\,\widehat{T}_{fi}(\Omega) \tag{36}$$

One may consider again the isoscalar $\pi\pi$ scattering in S-wave below 1.0 GeV. In terms of the phase-shift δ, the invariant amplitude is given by, from (18),

$$\widehat{T} = \frac{1}{\rho} e^{i\delta} \sin\delta \tag{37}$$

and when substituted into (31) the cross section (19) results. These expressions are very familiar, and they demonstrate clearly the interplay between the phase shifts, the invariant amplitudes and the cross sections.

One can similarly define the invariant analogue of the K-matrix through

$$K = \{\rho^\dagger\}^{\frac{1}{2}}\ \widehat{K}\ \{\rho\}^{\frac{1}{2}} \tag{38}$$

If $\widehat{K}$-matrix is taken to be real and symmetric, then K is Hermitian and unitarity is preserved, even when an element of ρ becomes imaginary below the threshold. From (12) one sees that

$$\widehat{K}^{-1} = \widehat{T}^{-1} + i\rho \tag{39}$$

which leads to

$$\widehat{T} = \widehat{K} + i\widehat{K}\rho\widehat{T} = \widehat{K} + i\widehat{T}\rho\widehat{K} \tag{40}$$

and

$$\widehat{T}\rho\widehat{K} = \widehat{K}\rho\widehat{T} \tag{41}$$

Solving for $\widehat{T}$, one obtains

$$\widehat{T} = \widehat{K}(I - i\rho\widehat{K})^{-1} = (I - i\widehat{K}\rho)^{-1}\widehat{K} \tag{42}$$

Note that $\widehat{K}$ and ρ do not commute. The Lorentz-invariant T-matrix is then given by

$$\widehat{T} = \frac{1}{1 - \rho_1\rho_2\widehat{D} - i(\rho_1\widehat{K}_{11} + \rho_2\widehat{K}_{22})} \begin{pmatrix} \widehat{K}_{11} - i\rho_2\widehat{D} & \widehat{K}_{12} \\ \widehat{K}_{21} & \widehat{K}_{22} - i\rho_1\widehat{D} \end{pmatrix} \tag{43}$$

where

$$\widehat{D} = \widehat{K}_{11}\widehat{K}_{22} - \widehat{K}_{12}^2 \tag{44}$$

Defining $\widehat{K}^{-1} = \widehat{U}$, one finds in addition

$$\widehat{T} = \frac{\begin{pmatrix} \widehat{U}_{22} - i\rho_2 & -\widehat{U}_{12} \\ -\widehat{U}_{21} & \widehat{U}_{11} - i\rho_1 \end{pmatrix}}{(\widehat{U}_{22} - i\rho_2)(\widehat{U}_{11} - i\rho_1) - \widehat{U}_{12}^2} \tag{45}$$

It was shown by Cahn and Landshoff[2] that the Flatté formula[6] may be derived as a special limiting case of the $\widehat{U}$-matrix. Let the channels 1 and 2 stand for $\pi\eta$ and $K\bar{K}$. Define

$$\begin{aligned} \widehat{U}_{22} &= \lambda g_1^2 + \frac{m_0^2}{2g_2^2} \\ \widehat{U}_{11} &= \lambda g_2^2 + \frac{m_0^2}{2g_1^2} \\ -\widehat{U}_{12} &= \lambda g_1 g_2 + \frac{m^2}{2g_1g_2} \end{aligned} \tag{46}$$

where g_i and m_0 are parameters in the problem, and let λ go to infinity. Then one finds

$$\widehat{T} = \frac{\begin{pmatrix} g_1^2 & g_1 g_2 \\ g_1 g_2 & g_2^2 \end{pmatrix}}{m_0^2 - m^2 - i(\rho_1 g_1^2 + \rho_2 g_2^2)} \tag{47}$$

This is the familiar Flatté formula. Note that the g_i are given in units of energy and for a resonance in S-wave it is also a constant; one may therefore define unitless constants γ_i via

$$g_i = \gamma_i \sqrt{m_0 \Gamma_0} \tag{48}$$

where Γ_0 is a constant playing the role of 'width' in the problem. Then, one may write

$$\widehat{T} = \frac{m_0 \Gamma_0}{m_0^2 - m^2 - i m_0 \Gamma_0 (\rho_1 \gamma_1^2 + \rho_2 \gamma_2^2)} \begin{pmatrix} \gamma_1^2 & \gamma_1 \gamma_2 \\ \gamma_1 \gamma_2 & \gamma_2^2 \end{pmatrix} \tag{49}$$

with the normalization condition

$$\gamma_1^2 + \gamma_2^2 = 1 \tag{50}$$

A second derivation of this formula is given in the next section, with further comments on the mass and the width.

4 Resonances in K-matrix Formalism

Resonances should appear as a sum of poles in the K-matrix. In the approximation of resonance domination for the amplitudes, one has therefore

$$K_{ij} = \sum_\alpha \frac{g^*_{\alpha i}(m) g_{\alpha j}(m)}{m_\alpha^2 - m^2} \tag{51}$$

and

$$\widehat{K}_{ij} = \sum_\alpha \frac{g^*_{\alpha i}(m) g_{\alpha j}(m)}{(m_\alpha^2 - m^2)\sqrt{\rho_i^* \rho_j}} \tag{52}$$

where the sum on α goes over the number of resonances with masses m_α, and the residue functions (expressed in units of energy) are given by

$$g^2_{\alpha i}(m) = m_\alpha \Gamma_{\alpha i}(m) \tag{53}$$

where $g_{\alpha i}(m)$ is real (but it could be negative) above the threshold for channel i. The width $\Gamma_\alpha(m)$ is

$$\Gamma_\alpha(m) = \sum_i \Gamma_{\alpha i}(m) \tag{54}$$

for each resonance α.

Consider now a resonance α coupling to n open two-body channels, i.e. the mass m_α is above the threshold of all the two-body channels. The partial widths may be given an expression

$$\Gamma_{\alpha i}(m) = \frac{g^2_{\alpha i}(m)}{m_\alpha} = \gamma^2_{\alpha i} \Gamma^0_\alpha B^2_{\alpha i}(m) \rho_i \tag{55}$$

and the residue function by

$$g_{\alpha i}(m) = \gamma_{\alpha i}\sqrt{m_\alpha \Gamma^0_\alpha}\ B_{\alpha i}(m)\sqrt{\rho_i} \tag{56}$$

where $B(m)$'s are the barrier factors

$$B_{\alpha i}(m) = \left[\frac{q_i(m)}{q_i(m_\alpha)}\right]^\ell \tag{57}$$

in terms of the breakup momentum in channel i and the orbital angular momentum ℓ. The γ's are real constants (but they can be negative) and may be given the normalization

$$\sum_i \gamma^2_{\alpha i} = 1 \tag{58}$$

In practice, it is probably better to avoid this normalization condition by using the parameters

$$g^0_{\alpha i} = \gamma_{\alpha i}\sqrt{m_\alpha \Gamma^0_\alpha} \tag{59}$$

as variables in the fit [see also (48)]. The residue function is then given by

$$g_{\alpha i}(m) = g^0_{\alpha i} B_{\alpha i}(m)\sqrt{\rho_i} \tag{60}$$

The observed total width $\tilde{\Gamma}_\alpha$ and the observed partial width $\tilde{\Gamma}_{\alpha i}$ are given by

$$\tilde{\Gamma}_\alpha = \sum_i \tilde{\Gamma}_{\alpha i} = \Gamma_\alpha(m_\alpha) = \Gamma^0_\alpha \sum_i \gamma^2_{\alpha i}\ \rho_i(m_\alpha) \tag{61}$$

From these one finds

$$\begin{aligned}
\Gamma^0_\alpha &= \sum_i \frac{\tilde{\Gamma}_{\alpha i}}{\rho_i(m_\alpha)} \\
\gamma^2_{\alpha i} &= \frac{\tilde{\Gamma}_{\alpha i}}{\Gamma^0_\alpha \rho_i(m_\alpha)} \\
g^0_{\alpha i} &= \sqrt{\frac{m_\alpha \tilde{\Gamma}_{\alpha i}}{\rho_i(m_\alpha)}}
\end{aligned} \tag{62}$$

In the limit in which the masses of the decay particles can be neglected compared to m_α, one has $\Gamma_\alpha(m_\alpha) \simeq \Gamma^0_\alpha$. In terms of the γ's and g^0's, the invariant K-matrix now has a simpler form

$$\begin{aligned}
\widehat{K}_{ij} &= \sum_\alpha \frac{\gamma_{\alpha i}\gamma_{\alpha j} m_\alpha \Gamma^0_\alpha B_{\alpha i}(m) B_{\alpha j}(m)}{m^2_\alpha - m^2} \\
&= \sum_\alpha \frac{g^0_{\alpha i} g^0_{\alpha j} B_{\alpha i}(m) B_{\alpha j}(m)}{m^2_\alpha - m^2}
\end{aligned} \tag{63}$$

Here one allows for the possibility that γ's and g^0's can be negative.

Consider now an isovector P-wave $\pi\pi$ scattering at or near the ρ mass. Then the elastic scattering amplitude at the $\pi\pi$-mass m is given by

$$K = \frac{m_0 \Gamma(m)}{m_0^2 - m^2} = \tan\delta \tag{64}$$

where m_0 is the mass of the ρ and δ is the usual phase shift. The mass-dependent width is given by

$$\Gamma(m) = \tilde{\Gamma}_0 \left(\frac{m_0}{m}\right) \left(\frac{q}{q_0}\right)^3 \tag{65}$$

where $\tilde{\Gamma}_0$ is the observed ρ width and q (q_0) is the $\pi\pi$ break-up momentum for the $\pi\pi$ mass m (m_0). Neglecting the angular dependence of the amplitude, one obtains

$$T = e^{i\delta} \sin\delta = \left[\frac{m_0 \tilde{\Gamma}_0}{m_0^2 - m^2 - i m_0 \Gamma(m)}\right] \left(\frac{q}{q_0}\right)^2 \left[\left(\frac{m_0}{m}\right)\left(\frac{q}{q_0}\right)\right] \tag{66}$$

The first bracket in (66) contains the usual Breit-Wigner form and the last bracket expresses the two-body phase-space factor. Note that the phase-space factor is absent in the Lorentz-invariant amplitude $\widehat{T}$ given by (26). The q^2 dependence of the amplitude reflects the fact that both the initial and the final $\pi\pi$ systems are in P-wave. The normalization for the transition amplitude has been chosen such that

$$T = +i \quad \text{and} \quad \widehat{T} = \frac{+i}{\rho} \quad \text{at} \quad m = m_0 \tag{67}$$

It is seen that the invariant amplitude $\widehat{T}$ is not normalized to 1 but to ρ^{-1}. It is for this reason that the Argand diagram is usually plotted with T and not $\widehat{T}$.

Consider again a $\pi\pi$ scattering at mass m. But suppose there exist two resonances with masses m_a and m_b coupling to the isoscalar S-wave channel. The prescription for the K-matrix in this case is that

$$K = \frac{m_a \Gamma_a(m)}{m_a^2 - m^2} + \frac{m_b \Gamma_b(m)}{m_b^2 - m^2} \tag{68}$$

i.e. the resonances are summed in the K-matrix. The mass-dependent widths are given by

$$\Gamma_\alpha(m) = \tilde{\Gamma}_\alpha^0 \left(\frac{m_a}{m}\right) \left(\frac{q}{q_\alpha}\right) \tag{69}$$

where $\alpha = a$ or $\alpha = b$ and $\tilde{\Gamma}_a^0$ and $\tilde{\Gamma}_b^0$ are the two observed widths in the problem. q_α is the $\pi\pi$ breakup momentum at $m = m_\alpha$. If m_a and m_b are far apart relative to the widths, then K is dominated either by the first or the second resonance depending on whether m is near m_a or m_b. The transition amplitude reduces to

$$T \simeq \left[\frac{m_\alpha \tilde{\Gamma}_\alpha^0}{m_\alpha^2 - m^2 - i m_\alpha \Gamma_\alpha(m)}\right] \left[\left(\frac{m_\alpha}{m}\right)\left(\frac{q}{q_\alpha}\right)\right] \tag{70}$$

for $m \simeq m_\alpha$. If one assumes that the two resonances dominate the region between m_a and m_b, then the total amplitude is given merely by the sum, i.e.

$$\begin{aligned} T &\simeq \left[\frac{m_a \tilde{\Gamma}_a^0}{m_a^2 - m^2 - i m_a \Gamma_a(m)}\right] \left[\left(\frac{m_a}{m}\right)\left(\frac{q}{q_a}\right)\right] \\ &+ \left[\frac{m_b \tilde{\Gamma}_b^0}{m_b^2 - m^2 - i m_b \Gamma_b(m)}\right] \left[\left(\frac{m_b}{m}\right)\left(\frac{q}{q_b}\right)\right] \end{aligned} \tag{71}$$

This provides a justification for the 'addition rule' of the Breit-Wigner forms within the K-matrix formalism. In the limit in which the two states have the same mass, i.e. $m_c \equiv m_a = m_b$, then the transition amplitude becomes

$$T = \frac{m_c[\Gamma_a(m) + \Gamma_b(m)]}{m_c^2 - m^2 - i m_c[\Gamma_a(m) + \Gamma_b(m)]} \tag{72}$$

This shows that the result is a single Breit-Wigner form but its total width is now the sum of the two individual widths.

As a next example one may take the case of a single resonance coupling to two channels. An example would be the isovector S-wave $a_0(980)$ coupling to the $\pi\eta$ (channel 1) and $K\bar{K}$ (channel 2) final states. Then the elements of the K-matrix are

$$\begin{aligned} K_{11} &= \frac{m_0 \Gamma_1(m)}{m_0^2 - m^2} \\ K_{22} &= \frac{m_0 \Gamma_2(m)}{m_0^2 - m^2} \\ K_{12} &= K_{21} = \frac{m_0 \sqrt{\Gamma_1(m)\Gamma_2(m)}}{m_0^2 - m^2} \end{aligned} \tag{73}$$

and

$$\Gamma_1(m) = \gamma_1^2\,\Gamma_0 \left(\frac{2q_1}{m}\right) = \gamma_1^2\,\Gamma_0 \rho_1 \quad \text{and} \quad \Gamma_2(m) = \gamma_2^2\,\Gamma_0 \left(\frac{2q_2}{m}\right) = \gamma_2^2\,\Gamma_0 \rho_2 \tag{74}$$

The nominal mass and width of the $a_0(980)$ are denoted by m_0 and Γ_0—they may be expressed in terms of the observed values, as shown later. The 'reduced' widths are denoted by γ_1^2 and γ_2^2, which are both unitless and satisfy

$$\gamma_1^2 + \gamma_2^2 = 1 \tag{75}$$

It is instructive to sketch a second derivation of the Flatté formula. The invariant K-matrix elements have constant numerators, as given by

$$\begin{aligned} \widehat{K}_{11} &= \frac{\gamma_1^2 m_0 \Gamma_0}{m_0^2 - m^2} \\ \widehat{K}_{22} &= \frac{\gamma_2^2 m_0 \Gamma_0}{m_0^2 - m^2} \\ \widehat{K}_{12} &= \widehat{K}_{21} = \frac{r \gamma_1 \gamma_2 m_0 \Gamma_0}{m_0^2 - m^2} \end{aligned} \tag{76}$$

Note that $r = \pm 1$ in the limit of factorization. Since $\widehat{D} = 0$, one readily finds from (43) the Flatté formula (49). The case $r^2 < 1$ has been explored; the resulting amplitudes contain a zero at $m = m_0$ in the numerator and thus they were deemed unacceptable.

The $a_0(980)$ appears as a 'regular' resonance in the $\pi\eta$ system (channel 1). Let the apparent mass and width in this channel be m_a and Γ_a. Then the denominator of (49) should appear as, for m near m_a,

$$m_a^2 - m^2 - i m_a \Gamma_a$$

in the resonance approximation. One finds, therefore,

$$\begin{aligned} m_0^2 &= m_a^2 - \left(\frac{\gamma_2}{\gamma_1}\right)^2 \left[\frac{|\rho_2(m_a)|}{\rho_1(m_a)}\right] m_a \Gamma_a \\ \Gamma_0 &= \frac{m_a \Gamma_a}{m_0 \rho_1(m_a) \gamma_1^2} \end{aligned} \tag{77}$$

in terms of the 'observed' mass m_a and width Γ_a in the $\pi\eta$ channel. Note that ρ_i's have been evaluated at $m = m_a$ where $\widehat{T}$ is expected to attain its maximum value. The above formulas give merely a good starting point; in practice one must vary m_0 and Γ_0 to fit the $\pi\eta$ spectrum. The ratio $(\gamma_2/\gamma_1)^2$ is an unknown (commonly fixed at the $SU(3)$ value of 1.5), but the shape of the square of the amplitudes depends only weakly on this value. Once the ratio is fixed, then γ_1^2 and γ_2^2 are fixed through the normalization condition (75).

5 Production of Resonances

So far one has considered s-channel resonances, or 'formation' of resonances, observed in the two-body scattering of the type $ab \to cd$. The K-matrix formalism can be generalized to cover the case of 'production' of resonances in more complex reactions. The key assumption is that the two-body system in the final state is an isolated one and that the two particles do not appreciably interact with the rest of the final state in the production process.

According to Aitchison[7], the production amplitude P should be transformed into F in the presence of two-body final state interactions, as follows:

$$F = (I - iK)^{-1} P = T K^{-1} P \tag{78}$$

Or, taking the invariant form, it may be written

$$\widehat{F} = (I - i\widehat{K}\rho)^{-1} \widehat{P} = \widehat{T}\widehat{K}^{-1}\widehat{P} \tag{79}$$

where $\widehat{P}$ characterizes production of a resonance and $\widehat{F}$ is the resulting invariant amplitude. It is emphasized that $\widehat{P}$ must have the same poles as those of the K-matrix. They are both column vectors, n-dimensional for an n-channel problem. If the K-matrix is given as a sum of the poles as in (51), then the corresponding P-vector is

$$P_i = \sum_\alpha \frac{\beta_\alpha^0 \, g_{\alpha i}(m)}{m_\alpha^2 - m^2} \tag{80}$$

and

$$\widehat{P}_i = \sum_\alpha \frac{\beta^0_\alpha \; g_{\alpha i}(m)}{(m^2_\alpha - m^2)\sqrt{\rho_i}} \tag{81}$$

where β^0_α is in general complex (expressed in units of energy), carrying the the production information of the resonance α. One has made an assumption here that there exists a single complex constant β^0_α for a given resonance α, independent of the decay channels. Under the assumption that the only thresholds present in the production process are those of the resonating two-body channel under study, one can put β^0_α real, since it does not have the thresholds.

It is often more convenient to rescale β^0's

$$\beta^0_\alpha = \beta_\alpha \sqrt{m_\alpha \Gamma^0_\alpha} \tag{82}$$

so that β's are unitless [see (59)]. Then the $\widehat{P}$-vectors read

$$\widehat{P}_i = \sum_\alpha \frac{\beta_\alpha \gamma_{\alpha i} \, m_\alpha \Gamma^0_\alpha B_{\alpha i}(m)}{m^2_\alpha - m^2} \tag{83}$$

where, once again, β's are complex and γ's are real. In the remaining part of this note one shall occasionally drop the barrier factor in the above formula, so that the numerators are now mass-independent, as follows:

$$\widehat{P}^0_i = \sum_\alpha \frac{\beta_\alpha \gamma_{\alpha i} \, m_\alpha \Gamma^0_\alpha}{m^2_\alpha - m^2} = \sum_\alpha \frac{\beta^0_\alpha \; g^0_{\alpha i}}{m^2_\alpha - m^2} \tag{84}$$

Then, the resulting production amplitude, defined by

$$\Delta(m) = (1 - i\widehat{K}\rho)^{-1} \widehat{P}^0 \tag{85}$$

may be considered a generalization of the familiar Breit-Wigner form. The complete production amplitude must of course contain the barrier factors as well as the angular dependence, which are traditionally considered 'external' to the Breit-Wigner form. It is instructive to work out the above formula in the case of a single resonance in a single channel. Then the $\widehat{K}\rho$ is given by (64) and

$$\widehat{P}^0 = \frac{\beta m_0 \Gamma_0}{m^2_0 - m^2}$$

so that

$$\Delta(m) = \frac{\beta m_0 \Gamma_0}{m^2_0 - m^2 - i m_0 \Gamma(m)} \tag{86}$$

This is exactly what one writes down for a Breit-Wigner form, except that one has multiplied by an arbitrary complex constant β. This provides a K-matrix justification of the traditional 'isobar' model. Note that the numerator is a constant, independent of m.

For a two-channel problem, the production amplitudes are

$$\widehat{F}_1 = \frac{\widehat{P}_1 - i\rho_2(\widehat{K}_{22}\widehat{P}_1 - \widehat{K}_{12}\widehat{P}_2)}{1 - \rho_1\rho_2\widehat{D} - i(\rho_1\widehat{K}_{11} + \rho_2\widehat{K}_{22})} \tag{87}$$

$$\widehat{F}_2 = \frac{\widehat{P}_2 - i\rho_1(\widehat{K}_{11}\widehat{P}_2 - \widehat{K}_{12}\widehat{P}_1)}{1 - \rho_1\rho_2\widehat{D} - i(\rho_1\widehat{K}_{11} + \rho_2\widehat{K}_{22})} \tag{88}$$

where $\widehat{D}$ is as given in (44). See Longacre[8] examples of the use of the $\widehat{P}$-vector approach.

Cahn and Landshoff[2] state that in some approximations the column vector $\widehat{Q} = \widehat{K}^{-1}\widehat{P}$ may be considered a constant in a given limited energy range. It is in general complex, but it can be real if β^0_α of $\widehat{P}$ is real. Then, one has

$$\widehat{F} = \widehat{T}\widehat{Q} \tag{89}$$

i.e. the two-body final-state interaction may be expressed as a product of the $\widehat{T}$-matrix and a constant column vector. The $\widehat{Q}$-vector is devoid of the threshold singularities (i.e., no dependence on ρ) and therefore depend in general on $s = m^2$ only. For a two-channel problem, one obtains

$$\widehat{F}_1 = \frac{(\widehat{K}_{11} - i\rho_2\widehat{D})\widehat{Q}_1 + \widehat{K}_{12}\widehat{Q}_2}{1 - i[\rho_1(\widehat{K}_{11} - i\rho_2\widehat{D}) + \rho_2\widehat{K}_{22}]} \tag{90}$$

$$\widehat{F}_2 = \frac{(\widehat{K}_{22} - i\rho_1\widehat{D})\widehat{Q}_2 + \widehat{K}_{12}\widehat{Q}_1}{1 - i[\rho_2(\widehat{K}_{22} - i\rho_1\widehat{D}) + \rho_1\widehat{K}_{11}]} \tag{91}$$

Au, Morgan and Pennington[3] also used this method with the $\widehat{Q}$-vector expanded as polynomials in m^2 (they assumed that $\widehat{Q}$ is real in their analysis of the double-Pomeron data).

Consider a single-channel problem, e.g. the isoscalar $\pi\pi$ system in S-wave below 1 GeV. Then, the K is simply given by (20) and one finds

$$\widehat{F} = e^{i\delta}\cos\delta\ \widehat{P} \tag{92}$$

In the $\widehat{Q}$-vector approach, one has similarly

$$\widehat{F} = \frac{1}{\rho}e^{i\delta}\sin\delta\ \widehat{Q} \tag{93}$$

In either case, the final-state interaction brings in a factor $e^{i\delta}$—this is the familiar Watson's theorem.

Consider next a single-resonance approximation to the two-channel problem. For concreteness one may wish to generalize the Flatté formula, i.e. the two-channel $a_0(980)$ formula appropriate for an arbitrary production process. Cahn and Landshoff[2] pointed out in fact that one can devise a number of models which differ fundamentally from the Flatté formula. It may be useful here to show yet another application of the K-matrix formalism. For the purpose one may start out with the expression for $\widehat{T}$ as given in (45) and set

$$\widehat{U}_{22} = x, \quad \widehat{U}_{11} = y \quad \text{and} \quad \widehat{U}_{12} = \widehat{U}_{21} = -z\left(\frac{m}{m_a}\right) \tag{94}$$

where x, y and z are unitless real constants to be determined, and then demand that the denominator approach

$$1 - \left(\frac{m}{m_a}\right)^2 - i\left(\frac{\Gamma_a}{m_a}\right)$$

for m near m_a. As before, m_a and Γ_a are the observed mass and width of the $a_0(980)$ in the $\pi\eta$ system (channel 1). Then one finds

$$\begin{aligned} y &= \left(\frac{m_a}{\Gamma_a}\right)\rho_1(m_a) \\ z^2 &= y\left\{x + |\rho_2(m_a)|\right\} \end{aligned} \tag{95}$$

Once again, the ρ_i's have been evaluated at $m = m_a$, and thus they are mere constants here. The parameter x is an additional variable left free to be determined by the data [the ratio y/x could be fixed to the $SU(3)$ value of 1.5]. Note that y is always positive but that x or z can be negative (but x should be greater than $-|\rho_2(m_a)|$).

If one takes the $\widehat{Q}$-vector approach, then the invariant amplitudes in the two channels may be written

$$\begin{aligned} \widehat{F}_1 &= \frac{(x - i\rho_2)\widehat{Q}_1 + z(m/m_a)\widehat{Q}_2}{xy - z^2(m/m_a)^2 - \rho_1\rho_2 - i(x\rho_1 + y\rho_2)} \\ \widehat{F}_2 &= \frac{(y - i\rho_1)\widehat{Q}_2 + z(m/m_a)\widehat{Q}_1}{xy - z^2(m/m_a)^2 - \rho_1\rho_2 - i(x\rho_1 + y\rho_2)} \end{aligned} \tag{96}$$

where $\widehat{Q}_1$ and $\widehat{Q}_2$ are additional parameters to be determined by the data. They are complex and may be expressed in general as polynomials in m^2. Note that these expressions are fundamentally different from the Flatté formula.

6 Multiple Resonances in Multi-Channels

As a concrete example one may consider two resonances coupling to two different channels. Consider then two isoscalar $J^{PC} = 0^{++}$ resonances m_a and m_b, with both masses around 1.0 GeV, coupling to $\pi\pi$ (channel 1) and $K\bar{K}$ (channel 2). The elements of the $\widehat{K}$-matrix are, assuming factorization for the residues,

$$\begin{aligned} \widehat{K}_{11} &= \gamma_{a1}^2\widehat{\Omega}_a^0(m) + \gamma_{b1}^2\widehat{\Omega}_b^0(m) \\ \widehat{K}_{22} &= \gamma_{a2}^2\widehat{\Omega}_a^0(m) + \gamma_{b2}^2\widehat{\Omega}_b^0(m) \\ \widehat{K}_{12} &= \widehat{K}_{21} = \gamma_{a1}\gamma_{a2}\widehat{\Omega}_a^0(m) + \gamma_{b1}\gamma_{b2}\widehat{\Omega}_b^0(m) \end{aligned} \tag{97}$$

$$\tag{98}$$

where

$$\widehat{\Omega}^0_a(m) = \frac{m_a\Gamma^0_a}{m_a^2 - m^2}, \quad \widehat{\Omega}^0_b(m) = \frac{m_b\Gamma^0_b}{m_b^2 - m^2} \tag{99}$$

Note that Γ^0_a and Γ^0_b are constants for scalar resonances coupling to two spinless particles. The normalizations are given by

$$\begin{aligned} \gamma_{a1}^2 + \gamma_{a2}^2 &= 1 \\ \gamma_{b1}^2 + \gamma_{b2}^2 &= 1 \end{aligned} \tag{100}$$

The $\widehat{P}^0$-vector can be written

$$\widehat{P}^0 = \begin{pmatrix} \beta_a\gamma_{a1}\widehat{\Omega}^0_a(m) + \beta_b\gamma_{b1}\widehat{\Omega}^0_b(m) \\ \beta_a\gamma_{a2}\widehat{\Omega}^0_a(m) + \beta_b\gamma_{b2}\widehat{\Omega}^0_b(m) \end{pmatrix} \tag{101}$$

where β_a and β_b are unitless complex constants specifying production of the resonances m_a and m_b. One can substitute these to the $\widehat{F}_i$'s given in (87) and (88), and the resulting amplitudes describe the production process involving two resonances in two channels.

An alternative method is to apply the $\widehat{Q}$-vector approach. Assume that $m_a \simeq m_b$. Then one has

$$(m_a^2 - m^2)(m_b^2 - m^2)\widehat{D} \simeq (\gamma_{a1}\gamma_{b2} - \gamma_{a2}\gamma_{b1})^2 m_a\Gamma^0_a m_b\Gamma^0_b = \alpha_1 \tag{102}$$

Note that α_1 is a constant, independent of m^2. One obtains, on the other hand,

$$\begin{aligned} (m_a^2 - m^2)(m_b^2 - m^2)\widehat{K}_{11} &= \alpha_2 + \alpha_3 m^2 \\ (m_a^2 - m^2)(m_b^2 - m^2)\widehat{K}_{22} &= \alpha_4 + \alpha_5 m^2 \\ (m_a^2 - m^2)(m_b^2 - m^2)\widehat{K}_{12} &= \alpha_6 + \alpha_7 m^2 \end{aligned} \tag{103}$$

where α_i's are real constants. One may further set, somewhat arbitrarily,

$$\begin{aligned} \widehat{Q}_1 &= \alpha_0 \\ \widehat{Q}_2 &= \alpha_8 + \alpha_9 m^{-2} \end{aligned} \tag{104}$$

A total of 10 parameters α_i has been introduced; α_0, α_8 and α_9 are complex in general, while the rest of the α's are real. Now the amplitudes $\widehat{F}_i$ can be cast into the form

$$\begin{aligned} \widehat{F}_1 &\rightarrow \frac{G_3(m)}{(m_a^2 - m^2)(m_b^2 - m^2) - i[\rho_1 G_1(m) + \rho_2 G_2(m)]} \\ \widehat{F}_2 &\rightarrow \frac{G_6(m)}{(m_a^2 - m^2)(m_b^2 - m^2) - i[\rho_2 G_4(m) + \rho_1 G_5(m)]} \end{aligned} \tag{105}$$

where, for $\widehat{F}_1$,

$$\begin{aligned} G_1(m) &= \alpha_2 + \alpha_3 m^2 - i\rho_2\alpha_1 \\ G_2(m) &= \alpha_4 + \alpha_5 m^2 \\ G_3(m) &= \alpha_2 + \alpha_3\lambda_1 m^2 - i\rho_2\alpha_1\lambda_2 + \lambda_3 m^{-2} \end{aligned} \tag{106}$$

and, for $\widehat{F}_2$,

$$\begin{aligned} G_4(m) &= \alpha_4 + \alpha_5 m^2 - i\rho_1\alpha_1 \\ G_5(m) &= \alpha_2 + \alpha_3 m^2 \\ G_6(m) &= \alpha_4 + \alpha_5\lambda_4 m^2 - i\rho_1\alpha_1\lambda_5 + (\lambda_6 - i\rho_1\lambda_7)m^{-2} \end{aligned} \quad (107)$$

Here λ_i's are complex constants. This is the formula used by David Bugg (attributed to Morgan and Pennington). Note that in this approach one has introduced a total of 12 parameters. Here the rationale is clear; the α_i for $i = 1, 5$ have been previously determined in the study of $\pi\pi$ and $K\bar{K}$ scattering amplitudes, and only the λ_i for $i = 1, 7$ are being varied to fit the data for 'production' of the $\pi\pi$ and $K\bar{K}$ systems.

Consider once again $\pi\pi$ (channel 1) and $K\bar{K}$ (channel 2) in S-wave. Au, Morgan and Pennington[3] gave a parameterization of the $\widehat{K}$-matrix in the following form:

$$\widehat{K}_{ij} = \left(\frac{s - s_0}{4m_K^2}\right)\left\{\sum_{p=1} \frac{f_i^{(p)} f_j^{(p)}}{(s_p - s)(s_p - s_0)} + \sum_{n=0} c_{ij}^{(n)} \left[\frac{s}{4m_K^2} - 1\right]^n\right\} \quad (108)$$

where $s = m^2$ and s_0 is the Adler zero. (K. Königsmann found a typographical error in Ref.[3]—the curly bracket in the above formula was missing.) Note that this parameterization represents a sum of a series of pole terms and a background term expressed as a polynomial in s. The fitted values can be read off their Table I. Their 'primary' fit (the first column in their Table I) consists of a single pole term ($s_1 = 0.9247$ GeV^2) and a polynomial for $n \leq 4$. Their fit covers a mass range of m from the $\pi\pi$ threshold up to 1.7 GeV, but considered reliable below 1.4 GeV. Their $\widehat{K}$-matrix can in principle be used to describe the production amplitude $\widehat{F}_i$ both in $\widehat{P}$- and $\widehat{Q}$-vector approaches. However, the $\widehat{P}$ vector must again contain a pole term and a background term consisting of a polynomial. A more economical way might be to take the $\widehat{F}_i$'s given in terms of two complex constants, $\widehat{Q}_1$ and $\widehat{Q}_2$ [see (90) and (91)]. Of course, it is more general to allow for energy dependence

$$\widehat{Q}_i = \sum_{n=0} d_i^{(n)} s^{n-k} \quad (109)$$

where $d_i^{(n)}$'s are complex constants to be determined by the data and k is an arbitrary integer. Note that $k = 1$ was adopted in (104).

As a final example, consider two isoscalar $J^{PC} = 2^{++}$ states, the $f_2(1270)$ and the $f_2(1565)$, both coupling to $\pi\pi$ (channel 1). Let the channel 2 [3] stand for the $f_2(1270)$ [$f_2(1565)$] decay into all other non-$\pi\pi$ channels. For simplicity one may further assume that the channels '2' and '3' are completely decoupled.

Then, the $\widehat{K}$-matrix takes on the form

$$\widehat{K} = \begin{pmatrix} \widehat{K}_{11} & \widehat{K}_{12} & \widehat{K}_{13} \\ \widehat{K}_{21} & \widehat{K}_{22} & \widehat{K}_{23} \\ \widehat{K}_{31} & \widehat{K}_{32} & \widehat{K}_{33} \end{pmatrix} \quad (110)$$

where

$$
\begin{aligned}
\widehat{K}_{11} &= \gamma_{a1}^2 \widehat{\Omega}_a + \gamma_{b1}^2 \widehat{\Omega}_b \\
\widehat{K}_{22} &= \gamma_{a2}^2 \widehat{\Omega}_a \\
\widehat{K}_{33} &= \gamma_{b3}^2 \widehat{\Omega}_b \\
\widehat{K}_{12} &= \widehat{K}_{21} = \gamma_{a1}\gamma_{a2}\widehat{\Omega}_a \\
\widehat{K}_{23} &= \widehat{K}_{32} = 0 \\
\widehat{K}_{13} &= \widehat{K}_{31} = \gamma_{b1}\gamma_{b3}\widehat{\Omega}_b
\end{aligned} \tag{111}
$$

The subscripts 'a' and 'b' stand for $f_2(1270)$ and $f_2(1565)$, respectively, and

$$
\widehat{\Omega}_a = \frac{m_a \Gamma_a^0 B_a^2(m)}{m_a^2 - m^2}, \qquad \widehat{\Omega}_b = \frac{m_b \Gamma_b^0 B_b^2(m)}{m_b^2 - m^2} \tag{112}
$$

where m_a and m_b are the masses. The barrier factors are given by

$$
B_a(m) = \left(\frac{q}{q_a}\right)^2, \qquad B_b(m) = \left(\frac{q}{q_b}\right)^2 \tag{113}
$$

so that the usual mass-dependent total widths $\Gamma_a(m)$ and $\Gamma_b(m)$ may be written

$$
\begin{aligned}
\Gamma_a(m) &= \Gamma_a^0 B_a^2(m)\rho_1(m) = \tilde{\Gamma}_a^0 \left(\frac{m_a}{m}\right) B_a^2(m) \left(\frac{q}{q_a}\right) \\
\Gamma_b(m) &= \Gamma_b^0 B_b^2(m)\rho_1(m) = \tilde{\Gamma}_b^0 \left(\frac{m_b}{m}\right) B_b^2(m) \left(\frac{q}{q_b}\right)
\end{aligned} \tag{114}
$$

where q_a and q_b are the breakup momenta in the $\pi\pi$ channel at $m = m_a$ and $m = m_b$, and the observed total widths are given by $\tilde{\Gamma}_a^0$ and $\tilde{\Gamma}_b^0$. Note that the coupling constants are constrained as follows:

$$
\begin{aligned}
\gamma_{a1}^2 + \gamma_{a2}^2 &= 1 \\
\gamma_{b1}^2 + \gamma_{b3}^2 &= 1
\end{aligned} \tag{115}
$$

One must note that an important simplifying assumption has been made here; the phase-space factors for the channels '2' and '3' are taken to be the same as that for channel '1' (the $\pi\pi$ system). As long as the channels '2' and '3' remain unspecified, one has in fact no choice. Note that the phase-space matrix with this assumption is given by $\rho = \rho_1 I$, i.e. it is proportional to the identity matrix. The $\widehat{P}^0$-vector can be written

$$
\widehat{P}^0 = \begin{pmatrix} \beta_a\gamma_{a1}\widehat{\Omega}_a^0 + \beta_b\gamma_{b1}\widehat{\Omega}_b^0 \\ \beta_a\gamma_{a2}\widehat{\Omega}_a^0 \\ \beta_b\gamma_{b3}\widehat{\Omega}_b^0 \end{pmatrix} \tag{116}
$$

where $\widehat{\Omega}_a^0$ and $\widehat{\Omega}_b^0$ are as defined in (99) [or (112) with the barrier factors removed], and β_a and β_b are complex constants specifying production of the states a and b. Note that the numerators of $\widehat{\Omega}_a^0$ and $\widehat{\Omega}_b^0$ have no dependence on m; it should be emphasized once more that the formula (84) is a generalization of the familiar Breit-Wigner form into the multi-channel problem. The overall amplitude one writes down must include the barrier factors, as they are not contained in the $\widehat{P}^0$-vector.

In the $\widehat{Q}$-vector approach, the generalized Breit-Wigner form reads

$$\Delta(m) = \widehat{T}\widehat{Q} = (1 - i\widehat{K}\rho)^{-1}\widehat{K}\widehat{Q} \tag{117}$$

One may take the Taylor series expansion $\widehat{Q}$ in terms of m^2. Once again, one must try both methods and choose that which gives the best fit to the data with the least number of parameters. However, to the extent that the β's may be taken as constants, the $\widehat{P}$-vector approach may be more efficient in general.

One may cast treatment of two isoscalar $J^{PC} = 2^{++}$ resonances, the $f_2(1270)$ and the $f_2(1565)$, into a two-channel problem as follows: let $\pi\pi$ be the channel 1, and let $\rho\rho$ be the channel 2. One assumes here that the $f_2(1565)$ has a substantial decay mode into $\rho\rho$, but the $f_2(1270)$ evidently decays very little into this channel. Then, the $\widehat{K}$-matrix takes on the form

$$\widehat{K} = \begin{pmatrix} \widehat{K}_{11} & \widehat{K}_{12} \\ \widehat{K}_{21} & \widehat{K}_{22} \end{pmatrix} \tag{118}$$

where

$$\begin{aligned} \widehat{K}_{11} &= \gamma_{a1}^2\widehat{\Omega}_{a1} + \gamma_{b1}^2\widehat{\Omega}_{b1} \\ \widehat{K}_{22} &= \gamma_{a2}^2\widehat{\Omega}_{a2} + \gamma_{b2}^2\widehat{\Omega}_{b2} \\ \widehat{K}_{12} &= \widehat{K}_{21} = \gamma_{a1}\gamma_{a2}\sqrt{\widehat{\Omega}_{a1}\widehat{\Omega}_{a2}} + \gamma_{b1}\gamma_{b2}\sqrt{\widehat{\Omega}_{b1}\widehat{\Omega}_{b2}} \end{aligned} \tag{119}$$

where γ's are real but can be negative. Note the normalization

$$\begin{aligned} \gamma_{a1}^2 + \gamma_{a2}^2 &= 1 \\ \gamma_{b1}^2 + \gamma_{b2}^2 &= 1 \end{aligned} \tag{120}$$

where $\gamma_{a1} = \pm 0.96$, $\gamma_{a2} = \pm 0.27$ for the $f_2(1270)$ [derived from the Particle Data Book], and γ_{b2}^2 for the $f_2(1565)$ should be large, e.g. 0.95. The subscripts 'a' and 'b' again stand for $f_2(1270)$ and $f_2(1565)$, respectively, and

$$\widehat{\Omega}_{a1} = \frac{m_a\Gamma_a^0 B_a^2(m)}{m_a^2 - m^2}$$

$$
\begin{aligned}
\widehat{\Omega}_{a2} &= \frac{m_a \Gamma_a^0 \bar{B}_a^2(m)}{m_a^2 - m^2} \\
\widehat{\Omega}_{b1} &= \frac{m_b \Gamma_b^0 B_b^2(m)}{m_b^2 - m^2} \\
\widehat{\Omega}_{b2} &= \frac{m_b \Gamma_b^0 \bar{B}_b^2(m)}{m_b^2 - m^2}
\end{aligned} \tag{121}
$$

The barrier factors $B_a(m)$ and $B_b(m)$ are as given in (113), and $\bar{B}_a(m)$ and $\bar{B}_b(m)$ are similar to $B_a(m)$ and $B_b(m)$, except that the breakup momenta are for the $\rho\rho$ system. They can be evaluated numerically in a straightforward manner. Let the effective masses for two ρ's be m_1 and m_2, and let m stand for the $\rho\rho$ effective mass. The four-body phase-space factor may be written (see Appendix B, Chung[4]), neglecting the angular dependence,

$$
d\varphi = \rho(m, m_1, m_2)\rho(m_1, \mu, \mu)\rho(m_2, \mu, \mu) ds_1 ds_2 \tag{122}
$$

where $\rho(m, m_1, m_2) = 2q(m, m_1, m_2)/m$ and q is the breakup momentum of a state of mass m into the two-body system with 'masses' m_1 and m_2. The pion masses are denoted by μ, and $s_1 = m_1^2$ and $s_2 = m_2^2$. Define Ψ containing the Breit-Wigner forms for the ρ's and the barrier factors for their decay, as follows:

$$
\Psi = \Delta^0(m_1) q(m_1, \mu, \mu) \Delta^0(m_2) q(m_2, \mu, \mu) \tag{123}
$$

where the Breit-Wigner forms are

$$
\Delta^0(m) = \frac{m_0 \Gamma_0}{m_0^2 - m^2 - i m_0 \Gamma(m)} \tag{124}
$$

The ρ mass and width are denoted by m_0 and Γ_0, and $\Gamma(m)$ is the mass-dependent width as given in (65). Then, the barrier factor for the $\rho\rho$ system is given by, for $\alpha = a$ or b,

$$
\bar{B}_\alpha^2(m) = \frac{\int d\varphi |\Psi \, q^2(m, m_1, m_2)/q^2(m_\alpha, m_1, m_2)|^2}{\int d\varphi |\Psi|^2} \tag{125}
$$

This formula is well defined for $m > 4\mu$; below this value one may analytically continue in the standard fashion, but one may also set it zero as m is now far away from the resonances masses.

The $\widehat{P}^0$-vector may be written

$$
\widehat{P}^0 = \begin{pmatrix} \beta_a \gamma_{a1} \widehat{\Omega}_a^0 + \beta_b \gamma_{b1} \widehat{\Omega}_b^0 \\ \beta_a \gamma_{a2} \widehat{\Omega}_a^0 + \beta_b \gamma_{b2} \widehat{\Omega}_b^0 \end{pmatrix} \tag{126}
$$

where $\widehat{\Omega}_a^0$ and $\widehat{\Omega}_b^0$ are as defined in (99). Substituting these into (84), one obtains the generalized Breit-Wigner form for the two-channel problem involving the two f_2's. The β's once again define production characteristics of the resonances 'a' and 'b.' Since the $f_2(1565)$ is seen only the $\bar{p}p$ annihilations, β_b is large compared to β_a. On the other hand, the $f_2(1565)$ branching ratio into $\pi\pi$ cannot be large; otherwise it would have been observed in previous analysis of the $\pi\pi$ channels. Therefore, γ_{b1} is likely to be small, but evidently one must have β_a comparable to $\beta_b\gamma_{b1}$, as both 'a' and 'b' are seen copiously in the $\pi\pi$ system from the $\bar{p}p$ annihilations.

Of course, one can also try out the $\widehat{Q}$-vector approach, with a polynomial expansion of the $\widehat{Q}$-vector in m^2.

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